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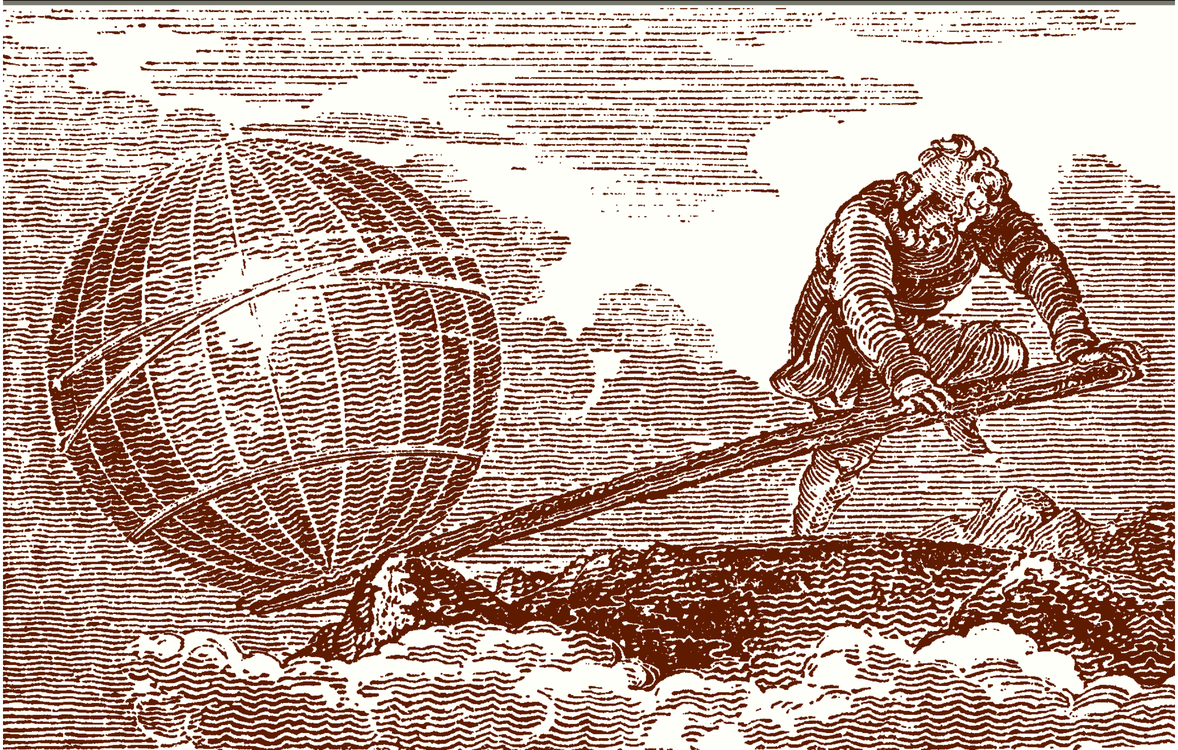

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CONTENTS **Vol. 3, 2010**

Larissa Borissova — De Sitter Bubble as a Model of the Observable Universe 3

Larissa Borissova — Gravitational Waves and Gravitational Inertial Waves According to the General Theory of Relativity 25

Patrick Marquet — Behaviour of the EGR Persistent Vacuum Field Following the Lichnérowicz Matching Conditions 71

Patrick Marquet — Geodesics and Finslerian Equations in the EGR Theory 90

Indranu Suhendro — A Hydrodynamical Geometrization of Matter and Chronometricity in General Relativity 102

De Sitter Bubble as a Model of the Observable Universe

Larissa Borissova

Abstract: Schwarzschild's metric of the space inside a sphere of incompressible liquid is taken under focus. We consider a particular case of the metric, where the surface of the liquid sphere meets the radius of gravitational collapse calculated for the mass. It is shown that, in this case, Schwarzschild's metric transforms into de Sitter's metric given that the cosmological λ-term of de Sitter's metric is positive (physical vacuum has positive density). Hence, in the state of gravitational collapse, the λ-field (physical vacuum) is equivalent to an ideal incompressible liquid whose density and pressure satisfy the equation of inflation (noting that positive density yields negative pressure). This result is then applied to the Universe as a whole, because it has mass, density, and radius such as those of a collapsar. The main conclusion of this study is: the Universe is a collapsar, whose internal space, being assumed to be a sphere of incompressible liquid, is a de Sitter space with positive density of physical vacuum.

Contents:

§1. Problem statement 3
§2. A sphere of incompressible liquid in the state of collapse as a model of the Universe 5
§3. Physically observable characteristics of a de Sitter space 8
§4. The cosmological λ-field is equivalent to an ideal incompressible liquid in the state of inflation 11
§5. Physically observable characteristics of a sphere of incompressible liquid 14
§6. Conditions of inhomogeneity and anisotropy 20
§7. Conclusion 23

§1. Problem statement. The main task of this study is to apply an extension of Schwarzschild's metric inside a sphere of incompressible liquid to cosmology. In other words, we will consider the Universe as a sphere of incompressible liquid. The extended Schwarzschild metric was obtained in my previous study [1]. It differs from the classical metric of the space inside a sphere of incompressible liquid, which was introduced in 1916 by Karl Schwarzschild [2], in the term g_{11} which allows space breaking. Schwarzschild omitted space breaks from consideration, which was a limitation imposed by him on the geometry. In contrast, we consider the geometry per se. This approach has already led us to some success: considering the Sun as a sphere of incompressible liquid, it was obtained that the break of g_{11} in the space of the Sun meets the Asteroid strip at the distance of the maximal concentration of substance [1].

The extended Schwarzschild metric has the form [1]

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\!\theta\, d\varphi^2\right), \tag{1.1}$$

where $\varkappa$ is Einstein's gravitational constant, ρ_0 is the density of the liquid, a is the radius of the liquid sphere, and r is the radial coordinate (whose origin is located at the centre of the sphere).

As was shown [1], the internal metric (1.1) of the liquid sphere being expressed through the density $\rho_0 = \frac{M}{V}$, the volume $V = \frac{4\pi a^3}{3}$, Einstein's constant $\varkappa = \frac{8\pi G}{c^2}$, and the Hilbert radius* $r_g = \frac{2GM}{c^2}$, takes the form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\!\theta\, d\varphi^2\right). \tag{1.2}$$

Assuming $r_g = a$ in the formula, we trivially arrive at the metric

$$ds^2 = \frac{1}{4}\left(1-\frac{r^2}{a^2}\right) c^2dt^2 - \frac{dr^2}{1-\frac{r^2}{a^2}} - r^2\left(d\theta^2 + \sin^2\!\theta\, d\varphi^2\right), \tag{1.3}$$

which is a particular case of de Sitter's metric. After the transformation of the time coordinate $\tilde{t} = \frac{1}{2}t$, this metric transforms into de Sitter's classical metric

$$ds^2 = \left(1-\frac{\lambda r^2}{3}\right) c^2dt^2 - \frac{dr^2}{1-\frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\!\theta\, d\varphi^2\right), \tag{1.4}$$

where $\lambda = \frac{3}{a^2} > 0$ in the particular case.

---

*This is the radius at which the field of a massive sphere (approximated as its centre of gravity — a mass-point) is in the state of gravitational collapse ($g_{00} = 0$). It is also known as the Schwarzschild radius, despite the fact that Karl Schwarzschild (1873–1916) never considered gravitational collapse in his papers of 1916 [2,3]. I refer to it as the *Hilbert radius* after David Hilbert (1862–1944) who considered it in 1917 [4], on the basis of the Schwarzschild mass-point solution [3].

Schwarzschild's metric inside a sphere of incompressible liquid is a solution of Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\,\varkappa T_{\alpha\beta}\,, \tag{1.5}$$

containing the energy-momentum tensor $T_{\alpha\beta}$ of ideal liquid, while the λ-term is assumed to be zero. At the same time, de Sitter's metric is a solution of Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = \lambda\, g_{\alpha\beta}\,, \tag{1.6}$$

where the energy-momentum tensor is zero, while the λ-term is nonzero. Since, as was shown, Schwarzschild's metric can be transformed into a particular case of de Sitter's metric, it would be interesting to find a correspondence between the energy-momentum tensor of incompressible liquid and the λ-term.

Proceeding from the formula for the energy-momentum tensor of ideal (non-viscous) incompressible liquid, we will see that the medium is equivalent to the λ-field (physical vacuum) under a particular condition, where the density and pressure satisfy the inflation equation $p=-\rho c^2$ (keeping in mind that positive density yields negative pressure).

§2. A sphere of an incompressible liquid in the state of collapse as a model of the Universe. Many models of the Universe are known, due to relativistic cosmology, as respective solutions to Einstein's equations. Initially, Albert Einstein believed that only stationary models of the Universe can be derived from the field equations. He therefore suggested a de Sitter space as a possible model of the Universe. This is a spherical space, filled with the λ-field (physical vacuum), and is described by de Sitter's metric. Then Alexander Friedmann proved that Einstein's equations can have non-stationary solutions. He obtained a class of solutions (models), which can be both stationary and non-stationary. The non-stationary Friedmann models can be expanding, compressing, or oscillating; the expanding models arise from a singular state, while the compressing and oscillating models can go through singular states during their evolution. All Friedmann models are homogeneous and isotropic. They are commonly accepted as the basis of the theory of a homogeneous, isotropic universe.

Already in 1966, Kyril Stanyukovich [5] had supposed that our Universe is a collapsar — an object in the state of gravitational collapse. He proceeded from a calculation, according to which an object, having mass and density equal to those of the Metagalaxy, has radius equal to the Hilbert radius calculated for the mass.

A collapsar means a static solution of Einstein's equations. Therefore, I suggest we go back to Einstein's initial suggestion of a de Sitter space, while taking Stanyukovich's calculation into account. Namely, I will consider the Universe as a collapsed sphere of incompressible liquid, described by the extended Schwarzschild metric (1.2), which can also be represented as a de Sitter space (1.3); thus the liquid gets the properties of physical vacuum (λ-field).

The term "gravitational collapse" is regularly used in connexion to the gravitational field derived from a spherical island of mass located in emptiness (for which Einstein's equations take the form $R_{\alpha\beta}=0$). The metric attributed to such spaces was introduced in 1916 by Karl Schwarzschild [3]. It is known as the Schwarzschild mass-point metric, or the mass-point metric in short

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{2.1}$$

where M is the island's mass (source of the field), while $r_g = \frac{2GM}{c^2}$ is the Hilbert radius calculated for the mass M. Once $r = r_g$, the time component of the fundamental metric tensor becomes zero ($g_{00}=0$): all the region under the surface $r = r_g$ around the massive island arrives at the state of gravitational collapse. If the island's radius r meets the surface of gravitational collapse, the island is obviously a collapsar.

The radius r_g is only defined by the mass of the massive island (source of the field). The radius r of the massive island itself comes from specific properties of the massive island itself. Therefore, in order for a massive island to be a collapsar, we should determine its properties so that its radius is equal to r_g. We would like to discover such a case.

Consider the space inside a sphere of incompressible liquid, whose radius is a. This case, first coined by Schwarzschild [2], arrives from Einstein's equations (1.5), where the energy-momentum tensor is attributed to ideal liquid (whose density is constant, $\rho = \rho_0 = const$)

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right) b^\alpha b^\beta - \frac{p}{c^2}\, g^{\alpha\beta}, \tag{2.2}$$

where p is the pressure of the liquid, while

$$b^\alpha = \frac{dx^\alpha}{ds}\,, \qquad b_\alpha b^\alpha = 1 \tag{2.3}$$

is the four-dimensional velocity vector, which characterizes the reference frame of an observer. The energy-momentum tensor should satisfy the conservation law

$$\nabla_\sigma T^{\alpha\sigma} = 0\,, \tag{2.4}$$

where ∇_σ is the symbol for generally covariant differentiation.

Assume, according to Stanyukovich [5], that the Universe is a collapsar. In addition to it, assume that the Universe is a sphere of incompressible ideal liquid, where galaxies play the rôle of molecules. In this case, the space of the Universe should be described by the extended Schwarzschild metric (1.2) with an additional condition

$$g_{00} = 0\,, \tag{2.5}$$

which points to the state of gravitational collapse. This condition, being applied to the metric (1.2), means that

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r}{a^3}}\right)^2 = 0\,. \tag{2.6}$$

It follows from (2.6) that a photometric radial distance $r = r_c$, at which the gravitational collapse occurs, is

$$r_c = a\sqrt{9-\frac{8a}{r_g}}\,, \tag{2.7}$$

thus r_c takes real values if the radius a of the liquid sphere is

$$a < \frac{9}{8}\,r_g\,, \tag{2.8}$$

while the radius of gravitational collapse becomes zero ($r_c = 0$) under the condition

$$a = \frac{9}{8}\,r_g = 1.125\,r_g\,. \tag{2.9}$$

Consider a particular case of (2.7), where the surface of the liquid sphere meets the Hilbert radius. In this case, we have

$$a = r_g \tag{2.10}$$

and, as follows from (2.7), the photometric distance also meets the collapse surface $r_c = r_g = a$. Hence:

> The internal field of a sphere of incompressible liquid in the state of gravitational collapse is equivalent to the external field of a collapsing mass-point as well.

Now, since Schwarzschild's metric of the space inside a sphere of incompressible liquid transforms into de Sitter's metric by the collapse condition and the condition $\lambda = \frac{3}{a^2}$, we arrive at the conclusion:

> Space inside a sphere of incompressible liquid, which is in the state of gravitational collapse, is described by de Sitter's metric, where the λ-term is $\lambda = \frac{3}{a^2}$.

All these can be applied to the Universe as a whole, because it has mass, density, and radius such as those of a collapsar. Therefore,

> The Universe is a collapsar, whose internal space, being assumed to be a sphere of incompressible liquid, is a de Sitter space with $\lambda = \frac{3}{a^2}$ (here a is the radius of the Universe).

§3. Physically observable characteristics of a de Sitter space. Herein, I consider physically observable properties of a de Sitter space, described by the metric (1.3), where $\lambda = \frac{3}{a^2}$. I use Zelmanov's mathematical apparatus of chronometric invariants [6–8]: chronometrically invariant quantities, being the respective projections of four-dimensional quantities onto the line of time and the three-dimensional section of an observer, are physically observable in his frame of reference.

According to the theory of chronometric invariants, the gravitational potential w and the linear velocity v_i of the rotation of space are

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right), \qquad v_i = \frac{c\,g_{0i}}{\sqrt{g_{00}}}\,, \qquad i = 1, 2, 3. \tag{3.1}$$

In both Schwarzschild's metric and de Sitter's metric, all the g_{0i} are zero, thus $v_i = 0$ (such a space does not rotate). Therefore, in these spaces, according to the chr.inv.-definition of the gravitational inertial force F_i and the angular velocity A_{ik} of the rotation of space [6–8],

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right) = 0\,, \tag{3.2}$$

$$F_i = \frac{c^2}{c^2 - \mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = -\frac{c^2}{2g_{00}}\frac{\partial g_{00}}{\partial x^i}\,. \tag{3.3}$$

Applying the formula for g_{00}, which follows from the metric (1.3) of the particular de Sitter space we are considering, we obtain

$$F_1 = \frac{c^2 r}{a^2 - r^2}\,, \qquad F_2 = F_3 = 0\,, \tag{3.4}$$

where $a^2 = \frac{3}{\lambda}$. Since we are considering a region of $r < a$, F_1 is positive. Hence, this is a gravitational inertial force of repulsion.

The chr.inv.-metric tensor, in a case of $v_i = 0$, takes the form

$$h_{ik} = -\,g_{ik} + \frac{1}{c^2}\, v_i v_k = -\,g_{ik}\,, \qquad h^{ik} = -\,g^{ik}, \qquad h^i_k = \delta^i_k\,, \tag{3.5}$$

where its substantial components for the metric (1.3) are

$$h_{11} = \frac{a^2}{a^2 - r^2}\,, \qquad h_{22} = r^2, \qquad h_{33} = r^2 \sin^2\!\theta\,, \tag{3.6}$$

$$h^{11} = \frac{a^2 - r^2}{a^2}\,, \qquad h^{22} = \frac{1}{r^2}\,, \qquad h^{33} = \frac{1}{r^2 \sin^2\!\theta}\,, \tag{3.7}$$

$$h = \det\|h_{ik}\| = \frac{a^2 r^4 \sin^2\!\theta}{a^2 - r^2}\,. \tag{3.8}$$

According to the chr.inv.-definition of the deformation of space,

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = 0\,, \qquad D^{ik} = -\,\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t} = 0\,, \tag{3.9}$$

where $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ is the chr.inv.-operator of differentiation with respect to time. Hence, such a space is free of deformation.

The chr.inv.-Christoffel symbols of the 1st kind and the 2nd kind

$$\Delta^k_{ij} = h^{km}\Delta_{ij,m} = \frac{1}{2}\,h^{km}\left(\frac{{}^*\partial h_{im}}{\partial x^j} + \frac{{}^*\partial h_{jm}}{\partial x^i} - \frac{{}^*\partial h_{ij}}{\partial x^m}\right), \tag{3.10}$$

are defined through $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\, v_i \frac{{}^*\partial}{\partial t}$, which is the chr.inv.-operator of differentiation with respect to the spatial coordinates. Their non-zero components of the metric (1.3) are

$$\Delta_{11,1} = \frac{a^2 r}{(a^2 - r^2)^2}\,, \qquad \Delta_{22,1} = -\,r\,, \qquad \Delta_{33,1} = -\,r\,\sin^2\!\theta\,, \tag{3.11}$$

$$\Delta_{12,2} = r\,, \qquad \Delta_{33,2} = -\,r^2 \sin\theta\cos\theta\,, \tag{3.12}$$

$$\Delta_{13,3} = r\,\sin^2\!\theta\,, \qquad \Delta_{23,3} = r^2 \sin\theta\cos\theta\,, \tag{3.13}$$

$$\Delta^1_{11} = \frac{r}{a^2 - r^2}\,, \quad \Delta^1_{22} = -\,\frac{(a^2 - r^2)\,r}{a^2}\,, \quad \Delta^1_{33} = -\,\frac{(a^2 - r^2)\,r}{a^2}\sin^2\!\theta\,, \tag{3.14}$$

$$\Delta^2_{12} = \frac{1}{r}\,, \qquad \Delta^2_{33} = -\,\sin\theta\cos\theta\,, \tag{3.15}$$

$$\Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{3.16}$$

The chr.inv.-curvature tensor C_{lkij},

$$C_{lkij} = H_{lkij} - \frac{1}{c^2}\big(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + \\ + A_{kl}D_{ij} + A_{li}D_{jk}\big)\,, \qquad (3.17)$$

which possesses all properties of the Riemann-Christoffel tensor in the spatial section, is determined through the chr.inv.-Schouten tensor

$$H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial \Delta_{kl}^{j}}{\partial x^i} - \frac{{}^*\partial \Delta_{il}^{j}}{\partial x^k} + \Delta_{kl}^{m}\Delta_{im}^{j} - \Delta_{il}^{m}\Delta_{km}^{j}\,. \qquad (3.18)$$

The contracted form $C_{lk} = C_{lki\cdot}^{\cdots i}$ of the chr.inv.-curvature tensor is

$$C_{lk} = H_{lk} - \frac{1}{c^2}\big(A_{kj}D_l^j + A_{lj}D_k^j + A_{kl}D\big)\,. \qquad (3.19)$$

In the absence of space rotation and deformation, which is specific to both Schwarzschild spaces and de Sitter spaces, H_{lkij} and C_{lkij} are the same.

For the metric (1.3), we obtain, according to the definition (3.18), the non-zero components of the chr.inv.-curvature tensor:

$$C_{121\cdot}^{\cdots 2} = C_{131\cdot}^{\cdots 3} = -\frac{1}{a^2 - r^2}\,, \qquad C_{232\cdot}^{\cdots 3} = -\frac{r^2}{a^2}\,, \qquad (3.20)$$

thus, respectively,

$$C_{1212} = -\frac{r^2}{a^2 - r^2}\,, \quad C_{1313} = -\frac{r^2\sin^2\theta}{a^2 - r^2}\,, \quad C_{2323} = -\frac{r^4\sin^2\theta}{a^2}\,, \quad (3.21)$$

and also, the non-zero components of the contracted tensor:

$$C_{11} = -\frac{2}{a^2 - r^2}, \qquad C_{22} = -\frac{C_{33}}{\sin^2\theta} = -\frac{2r^2}{a^2}\,. \qquad (3.22)$$

As a result, we obtain the chr.inv.-curvature (observable curvature) of the three-dimensional space (spatial section). It is

$$C = -\frac{6}{a^2} = const < 0\,, \qquad (3.23)$$

so a de Sitter space having the metric (1.3) is a space of constant negative three-dimensional curvature, where the curvature is inversely proportional to the square of the radius of the space.

These are the physically observable characteristics of a de Sitter space, which has the particular metric (1.3), where $\lambda = \frac{3}{a^2}$.

§4. The cosmological λ-field is equivalent to an ideal incompressible liquid in the state of inflation. When looking for an exact solution of Einstein's equations while taking a given distribution of matter (the energy-momentum tensor) into account, we should solve them in common with the law of conservation (2.4), which determines the distribution. As is known, de Sitter spaces are filled with λ-fields, thus they are described by the particular form (1.6) of Einstein's equations. On the other hand, as was shown earlier, a de Sitter space containing $\lambda = \frac{3}{a^2}$ is a particular case of a Schwarzschild space inside a sphere of incompressible liquid, wherein Einstein's equations have the form (1.5). Our task here is to find, by solving Einstein's equations and the equations of energy-momentum conservation, how the properties of ideal liquid are linked to the λ-field in this particular case.

We therefore consider the general form

$$R_{\alpha\beta} - \frac{1}{2}\,R g_{\alpha\beta} = -\,\varkappa\, T_{\alpha\beta} + \lambda\, g_{\alpha\beta} \tag{4.1}$$

of Einstein's equations, which covers both de Sitter spaces and Schwarzschild spaces.

According to the theory of chronometric invariants [6–8], the energy-momentum tensor has three observable chr.inv.-components (as well as any symmetric tensor of the 2nd rank):

$$\rho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}, \tag{4.2}$$

where ρ is the chr.inv.-density of the distributed matter, J^i is the chr.inv.-vector of the density of the momentum in the medium, U^{ik} is the chr.inv.-stress tensor.

Assume that the space is filled with an ideal (non-viscous) incompressible ($\rho = \rho_0 = const$) liquid. In this case, the energy-momentum tensor has the form (2.2), where the density and pressure of the liquid satisfy the equation of state

$$\rho c^2 = -\,p\,, \tag{4.3}$$

known as *the state of inflation*. Respectively, we obtain the chr.inv.-components of the energy-momentum tensor (2.2). They are

$$\rho = \rho_0\,, \qquad J^i = 0\,, \qquad U^{ik} = p h^{ik} = -\,\rho_0\, c^2 h^{ik}, \tag{4.4}$$

being derived from (2.2) through the condition

$$b^i = \frac{dx^i}{ds} = 0\,, \qquad i = 1, 2, 3, \tag{4.5}$$

which means that the observer accompanies his references. The first chr.inv.-component, $\rho = \rho_0$, means that the liquid is incompressible. The second chr.iv.-component, $J^i = 0$, means that the liquid does not contain flows of momentum. The third chr.inv.-component, $U^{ik} = ph^{ik}$, means that the obsever accompanies the medium. In other words, a regular observer rests with respect to the medium and its flows.

Chr.inv.-projections of Einstein's equations (4.1) has been obtained in the framework of the theory of chronometric invariants [6–8]. They are known as the Einstein chr.inv.-equations

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{jl} + A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2}F_jF^j = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \lambda c^2, \quad (4.6)$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}F_jA^{ij} = \varkappa J^i, \quad (4.7)$$

$$\frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D^j_k + A^{\cdot j}_{k\cdot}\right) + DD_{ik} + 3A_{ij}A^{\cdot j}_{k\cdot} + \\ + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_iF_k - c^2C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \lambda c^2 h_{ik}\,, \quad (4.8)$$

where $U = h^{ik}U_{ik}$ is the trace of the chr.inv.-stress tensor U_{ik}, while ${}^*\nabla_i$ is the symbol for chr.inv.-differentiation. The chr.inv.-components of the conservation law (2.4) have the form [6–8]

$$\frac{{}^*\partial\rho}{\partial t} + D\rho + \frac{1}{c^2}D_{ij}U^{ij} + {}^*\nabla_i J^i - \frac{2}{c^2}F_iJ^i = 0\,, \quad (4.9)$$

$$\frac{{}^*\partial J^k}{\partial t} + DJ^k + 2\left(D^k_i + A^{\cdot k}_{i\cdot}\right)J^i + {}^*\nabla_i U^{ik} - \frac{2}{c^2}F_iU^{ik} - \rho F^k = 0\,. \quad (4.10)$$

Take into account that for the metric (1.3)

$$D_{ik} = 0\,, \quad A_{ik} = 0\,, \quad J^i = 0\,, \quad U_{ik} = ph_{ik}\,, \quad U = 3p\,, \quad (4.11)$$

and the inflation state $\rho c^2 = -p$. Under these conditions, the Einstein chr.inv.-equations (4.6–4.8) take the form

$${}^*\nabla_j F^j - \frac{1}{c^2}F_jF^j = -\frac{\varkappa}{2}\left(\rho_0 c^2 + 3p\right) + \lambda c^2 = \left(\varkappa\rho_0 + \lambda\right)c^2, \quad (4.12)$$

$$\frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_iF_k - c^2C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho_0 c^2 - p\right)h_{ik} + \lambda c^2 = \left(\varkappa\rho_0 + \lambda\right)c^2. \quad (4.13)$$

We calculate

$$^*\nabla_j F^j = {}^*\nabla_1 F^1 = \frac{^*\partial F^1}{\partial x^1} + \frac{^*\partial \ln\sqrt{h}}{\partial x^1}\, F^1 = \frac{c^2(3a^2-2r^2)}{a^2(a^2-r^2)}\,, \tag{4.14}$$

$$^*\nabla_1 F_1 = \frac{^*\partial F_1}{\partial x^i} - \Delta^1_{11} F_1 = \frac{c^2}{a^2-r^2} + \frac{c^2 r^2}{(a^2-r^2)^2}\,, \tag{4.15}$$

$$^*\nabla_2 F_2 = -\Delta^1_{22} F_1 = \frac{c^2 r^2}{a^2}\,, \tag{4.16}$$

$$^*\nabla_3 F_3 = -\Delta^1_{33} F_1 = \frac{c^2 r^2 \sin^2\theta}{a^2}\,, \tag{4.17}$$

then substitute these, and also F_1, C_{11}, C_{22}, C_{33} calculated according to the formulae of §3, into the Einstein chr.inv.-equations (4.12–4.13). After algebra, we obtain that only one equation of the Einstein chr.inv.-equations remains non-vanishing:

$$\frac{3c^2}{a^2} = (\varkappa\rho_0 + \lambda)\, c^2. \tag{4.18}$$

Consider two formal cases for this equation, satisfying both the Schwarzschild metric and the particular de Sitter metric. Namely:

1) A case, where $T_{\alpha\beta} \neq 0$ and $\lambda = 0$. This means that the space is filled only with distributed matter (ideal incompressible liquid, in this case). Thus, we obtain, from the Einstein chr.inv.-equation (4.18), the density and pressure of the liquid

$$\rho_0 = \frac{3}{\varkappa a^2}\,, \qquad p = -\rho_0 c^2 = -\frac{3c^2}{\varkappa a^2} = const, \tag{4.19}$$

while the chr.inv.-equations of the conservation law (4.9–4.10) are satisfied as identities;

2) Another option is that of $T_{\alpha\beta} = 0$ and $\lambda \neq 0$. In this case, the space is filled only with physical vacuum (λ-field). Thus, the Einstein chr.inv.-equation (4.18) reduces to

$$\lambda = \frac{3}{a^2} > 0\,, \tag{4.20}$$

so the density and pressure of physical vacuum are expressed through the λ-term, according to the chr.inv.-equations of the conservation law (4.9–4.10), as

$$\rho_0 = \frac{\lambda}{\varkappa}\,, \qquad p = -\frac{\lambda c^2}{\varkappa} = const. \tag{4.21}$$

Therefore, since these two cases meet each other in the particular case under consideration, we arrive at the conclusion:

> The λ-field (physical vacuum), which fills a particular de Sitter space, where $\lambda = \frac{3}{a^2} > 0$, is equivalent to an ideal incompressible liquid in the state of inflation.

§5. Physically observable characteristics of a sphere of incompressible liquid. Here we compare the details of two different states of the space inside a sphere of incompressible liquid:

1) A regular state of the liquid sphere, where its radius a is much larger than the Hilbert radius r_g calculated for the mass ($a \gg r_g$). I refer to such an object as a *Schwarzschild bubble*, since its internal space is described by the Schwarzschild metric (1.2);
2) The liquid sphere is a collapsar — a body in the state of gravitational collapse. In this case, the surface of the sphere meets its Hilbert radius ($a = r_g$). I suggest that such an object should be referred to as a *de Sitter bubble*. This is because its internal space is described by the particular de Sitter metric (1.3).

First of all, we would like to point out numerous principal differences of this consideration from that according to the Schwarzschild mass-point metric utilized by most relativists when considering collapsars [9].

According to the mass-point metric (2.1), $g_{00} > 0$ in the space outside the collapsed surface ($r > r_g$), $g_{00} = 0$ on the surface ($r = r_g$), and $g_{00} < 0$ in the space inside it ($r < r_g$). Thus the signature condition $g_{00} > 0$ is violated inside gravitational collapsars. In order to restore the signature condition $g_{00} > 0$ inside collapsars, another metric is suggested: it is derived from the mass-point metric (2.1) by substitution of $r = c\tilde{t}$ and $ct = \tilde{r}$, thus space and time replace each other. As a result, the signature condition remains valid inside collapsars, but is violated in the regular space surrounding them [9]. Also, the mass-point metric does not specify the body's radius. In other words, we cannot recognize, without additional conditions, whether the object is a collapsar, or not.

By contrast, the signature condition $g_{00} > 0$ is satisfied everywhere inside a collapsar filled with incompressible liquid (Schwarzschild space) or physical vacuum (de Sitter's space). In addition to it, both metrics contain the radius of space. Thus, we can clearly recognize, from the metric itself, that the considered object is a collapsar ($a = r_g$). These are advantages of our approach.

Probably, there are many such objects in the Universe: consisting of a substance similar to ideal incompressible liquid, they may transform into collapsars at the final stage of their evolution, thus becoming de Sitter bubbles. These objects are hidden from observation, because, being collapsars, they never allow light to leave their internal space for the cosmos.

Let us derive a formula for the chr.inv.-vector of the gravitational inertial force from the metric (1.2). We obtain that just one (radial) component of the force is non-zero. It is

$$F_1 = -\frac{c^2 r_g r}{a^3} \frac{1}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} \,. \tag{5.1}$$

Since $r < a$ inside the sphere, $F_1 < 0$ therein. Hence, this is a force of attraction. It is $F_1 = 0$ at the centre of the sphere, and $F_1 \to -\infty$ on its surface (the surface of gravitational collapse).

Consider a regular case, where $a \gg r_g$. Expanding $\sqrt{1-\frac{r_g r^2}{a^3}}$ into series, while neglecting the high order terms, we obtain

$$\sqrt{1-\frac{r_g r^2}{a^3}} \approx 1 - \frac{r_g r^2}{2a^3} \,, \tag{5.2}$$

thus, once $r = a$, we have

$$\sqrt{1-\frac{r_g}{a}} \approx 1 - \frac{r_g}{2a} \,. \tag{5.3}$$

Substituting (5.2) into (5.1), we obtain

$$F_1 \approx -\frac{c^2 r_g r}{2a^3} = -\frac{GMr}{a^3} \,. \tag{5.4}$$

If $r = a$, we obtain a Newtonian gravitational force of attraction, which is $F_1 \approx -\frac{GM}{a^2}$.

It is easy to show that F_1 (5.1) by $a = r_g$ takes the form

$$F_1 = \frac{c^2 r}{a^2 - r^2} > 0 \,, \tag{5.5}$$

which is a non-Newtonian gravitational force of repulsion:

> The gravitational inertial force inside a regular sphere of incompressible liquid and in that in the state of being a collapsar has opposite signs. In a regular liquid sphere (Schwarzschild bubble), this is a Newtonian gravitational force of attraction. In a liquid sphere which is a collapsar (de Sitter bubble), this is a repulsing non-Newtonian gravitational force.

The pressure inside a regular liquid sphere (1.2) is formulated as [1]

$$p = \rho_0 c^2 \, \frac{\sqrt{1-\frac{r_g r^2}{a^3}} - \sqrt{1-\frac{r_g}{a}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{5.6}$$

so $p > 0$ under $a \gg r_g$. Once $a = r_g$, the pressure takes the form

$$p = -\rho_0 c^2 = const, \tag{5.7}$$

thus the medium is in the state of inflation. Since $\rho_0 > 0$, we obtain that $p < 0$ inside de Sitter bubbles. So, we conclude:

> The pressure is positive in a regular sphere of incompressible liquid. It is negative in a liquid sphere, which is a collapsar.

Consider the chr.inv.-curvature tensor C_{lkij} for the metric (1.2). First, we obtain the components of the chr.inv.-metric tensor

$$h_{11} = \frac{1}{1-\frac{r_g r^2}{a^3}}\,, \quad h_{22} = r^2, \quad h_{33} = r^2 \sin^2\theta\,, \tag{5.8}$$

$$h^{11} = 1 - \frac{r_g r^2}{a^3}\,, \quad h^{22} = \frac{1}{r^2}\,, \quad h^{33} = \frac{1}{r^2 \sin^2\theta}\,, \tag{5.9}$$

$$h = \det\|h_{ik}\| = \frac{r^4 \sin^2\theta}{1-\frac{r_g r^2}{a^3}}\,, \tag{5.10}$$

and the chr.inv.-Christoffel symbols

$$\Delta_{11,1} = \frac{r_g r}{a^3} \frac{1}{\left(1-\frac{r_g r^2}{a^3}\right)^2}\,, \quad \Delta_{22,1} = -r\,, \quad \Delta_{33,1} = -r \sin^2\theta, \tag{5.11}$$

$$\Delta_{12,2} = r\,, \qquad \Delta_{33,2} = -r^2 \sin\theta\cos\theta\,, \tag{5.12}$$

$$\Delta_{13,3} = r \sin^2\theta\,, \qquad \Delta_{23,3} = r^2 \sin\theta\cos\theta\,, \tag{5.13}$$

$$\left.\begin{array}{c} \Delta^1_{11} = \dfrac{r_g r}{a^3\left(1-\frac{r_g r}{a^3}\right)}\,, \quad \Delta^1_{22} = -r\left(1-\dfrac{r_g r^2}{a^3}\right) \\ \Delta^1_{33} = -r\left(1-\dfrac{r_g r^2}{a^3}\right)\sin^2\theta \end{array}\right\}, \tag{5.14}$$

$$\Delta^2_{12} = \frac{1}{r}\,, \quad \Delta^2_{33} = -\sin\theta\cos\theta\,, \tag{5.15}$$

$$\Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{5.16}$$

Then we obtain the non-zero components of C_{iklj}

$$C_{1212} = \frac{C_{1313}}{\sin^2\theta} = -\,\frac{r_g}{a^3}\,\frac{r^2}{1-\frac{r_g r^2}{a^3}}\,, \quad C_{2323} = -\,\frac{r_g}{a^3}\,r^4\sin^2\theta\,, \tag{5.17}$$

which coincide with those (3.21) obtained for the particular de Sitter metric (1.3) by the condition $a = r_g$, i.e. when the liquid sphere is a collapsar. Contracting these with h_{ik}, we obtain the non-zero components of the contracted chr.inv.-curvature tensor

$$C_{11} = -\frac{2r_g}{a^3}\frac{1}{1-\frac{r_g r^2}{a^3}}\,, \qquad C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2r_g r^2}{a^3}\,, \tag{5.18}$$

and also the chr.inv.-curvature scalar (observable curvature of the three-dimensional space)

$$C = -\frac{6\,r_g}{a^3} = const < 0\,, \tag{5.19}$$

which coincides, by the collapse condition $a = r_g$, with the respective values (3.22) and (3.23), obtained for the particular de Sitter metric (1.3). Hence, a Schwarzschild space with the metric (1.2) has a constant negative observable (three-dimensional) curvature space.

It should be noted that, as one may find in any textbook of the theory of relativity and relativistic cosmology, de Sitter spaces are constant curvature spaces, while Schwarzschild spaces are not. This commonly accepted terminology is based on the four-dimensional curvature K. The observable (three-dimensional) chr.inv.-curvature C is calculated in another way; it is linked to K only in constant curvature spaces such as de Sitter spaces (see §5.3 in [10], for details). Thus,

> In a de Sitter space, the four-dimensional curvature K and observable (three-dimensional) curvature C are constants. A Schwarzschild space, which is inside a sphere of incompressible liquid, has a variable four-dimensional curvature K and a constant observable (three-dimensional) curvature C.

The observable three-dimensional curvature of such a Schwarzschild bubble has a radius $\Re$, which, coming from the relation

$$C = -\frac{6\,r_g}{a^3} = \frac{1}{\Re^2}\,, \tag{5.20}$$

which is obvious for a liquid sphere, is imaginary

$$\Re = \frac{ia\sqrt{a}}{\sqrt{6\,r_g}}\,. \tag{5.21}$$

Respectively, the observable curvature radius of a de Sitter bubble, according to the formula of C (3.23), is imaginary as well

$$\Re = \frac{ia}{\sqrt{6}} \,. \tag{5.22}$$

Now, we consider the four-dimensional curvature of spaces with the metrics (1.2) and (1.3). The Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ has three chr.inv.-components [6–8]

$$X^{ik} = -\,c^2\,\frac{R_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \quad Y^{ijk} = -\,c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \quad Z^{iklj} = c^2 R^{iklj}, \tag{5.23}$$

which, according to the theory of chronometric invariants, are generally formulated through the chr.inv.-characteristics of the space of reference of an observer as follows (the indices in X^{ik}, Y^{ijk}, Z^{iklj} have been lowered here by the chr.inv.-metric tensor h_{ik}):

$$X_{ik} = \frac{{}^*\partial D_{ik}}{\partial t} - \left(D_i^l + A_{i\cdot}^{\cdot l}\right)\left(D_{kl} + A_{kl}\right) + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2} F_i F_k \,, \tag{5.24}$$

$$Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2} A_{ij} F_k \,, \tag{5.25}$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2 C_{iklj} \,. \tag{5.26}$$

Because $A_{ik}=0$ and $D_{ik}=0$ for both the metric (1.2) and the metric (1.3), the formulae (5.24–5.26) take a simplified form, which is

$$X_{ik} = \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2} F_i F_k \,, \tag{5.27}$$

$$Y_{ijk} = 0 \,, \tag{5.28}$$

$$Z_{iklj} = -\,c^2 C_{iklj} \,. \tag{5.29}$$

In particular, we see that, in the metrics (1.2) and (1.3) (that is, in the space inside a Schwarzschild bubble or a de Sitter bubble respectively), the spatial observable projection Z_{iklj} of the Riemann-Christoffel curvature tensor (its distribution along the three-dimensional spatial section) is proportional to the chr.inv.-curvature tensor C_{iklj}, taken with the opposite sign:

> The observable distribution of the Riemann-Christoffel curvature tensor inside both a Schwarzschild bubble and a de Sitter bubble is the same as that of the observable three-dimensional curvature tensor therein, but has the opposite sign.

Let us calculate X_{ik} for the metric (1.2). This is the chr.inv.-projection of the Riemann-Christoffel curvature tensor onto the line of time of an observer. Its formula (5.27) can be re-written, expanding the symbol of the chr.inv.-differentiation, in the form

$$X_{ik} = \frac{1}{2}\left(\frac{{}^*\partial F_i}{\partial x^k} + \frac{{}^*\partial F_k}{\partial x^i}\right) - \Delta^m_{ik} F_m - \frac{1}{c^2} F_i F_k \,, \tag{5.30}$$

thus we obtain nonzero components of X_{ik}. They are

$$X_{11} = -\frac{c^2 r_g}{a^3}\,\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{5.31}$$

$$X_{22} = -\frac{c^2 r_g}{a^3}\,\frac{r^2\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{5.32}$$

$$X_{33} = -\frac{c^2 r_g}{a^3}\,\frac{r^2 \sin^2\theta\,\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{5.33}$$

Assuming $a = r_g$ that means the metric (1.3), we obtain the same spatial components inside a de Sitter bubble

$$X_{11} = \frac{c^2}{a^2 - r^2}\,, \quad X_{22} = \frac{c^2 r^2}{a^2}\,, \quad X_{33} = \frac{c^2 r^2 \sin^2\theta}{a^2}\,. \tag{5.34}$$

We see that all non-zero components of X_{ik} are negative in Schwarzschild bubbles, while they are positive in de Sitter bubbles.

Let us compare the formulae of X_{11} with the respective formulae of F_1 in Schwarzschild bubbles (5.1) and in de Sitter bubbles (3.4). We see that in both cases they are connected by the relation

$$F_1 = r\,X_{11}\,, \tag{5.35}$$

thus we arrive at the following important result:

> The time observable component of the Riemann-Christoffel curvature tensor has the same numerical value, but opposite signs in the spaces of a Schwarzschild bubble and a de Sitter bubble. Newtonian gravitational forces of attraction in Schwarzschild bubbles and non-Newtonian gravitational forces of repulsion in de Sitter bubbles are only due to the time observable component of the curvature tensor.

§6. Conditions of inhomogeneity and anisotropy. According to the theory of chronometric invariants [6,8], the conditions of homogeneity have the form

$$\left.\begin{array}{llll} {}^*\nabla_j F_i = 0\,, & {}^*\nabla_j A_{ik} = 0\,, & {}^*\nabla_j D_{ik} = 0\,, & {}^*\nabla_j C_{ik} = 0 \\ \dfrac{{}^*\partial\rho}{\partial x^i} = 0\,, & \dfrac{{}^*\partial p}{\partial x^i} = 0\,, & {}^*\nabla_j \beta_{ik} = 0\,, & {}^*\nabla_j q_i = 0 \end{array}\right\}, \qquad (6.1)$$

where $\beta_{ik} = \alpha_{ik} - \frac{1}{3}\,\alpha h_{ik}$ is the anisotropic part of the viscous stress tensor α_{ik}, $\alpha = \alpha_n^n$, and $q_i = c^2 J_i$ is the chr.inv.-vector of the density of the flow of energy. In other words, once a three-dimensional spatial section satisfies the conditions (6.1), it is homogeneous from the point of view of an observer located in it. The conditions of isotropy are

$$F_i = 0\,, \quad A_{ik} = 0\,, \quad \Pi_{ik} = 0\,, \quad \Sigma_{ik} = 0\,, \quad \beta_{ik} = 0\,, \quad q_i = 0\,, \qquad (6.2)$$

where $\Pi_{ik} = D_{ik} - \frac{1}{3}\,Dh_{ik}$ and $\Sigma_{ik} = C_{ik} - \frac{1}{3}\,Ch_{ik}$ characterize, respectively, the anisotropy of the deformation and curvature of space. If a spatial section satisfies the conditions (6.2), it is observed as isotropic.

Let us apply the physical conditions of the metrics (1.2) and (1.3) to the conditions of homogeneity and isotropy. In both these metrics, $A_{ik} = 0$ and $D_{ik} = 0$. Also, we should take into account that $\rho_0 = const$, $\beta_{ik} = 0$, and $J_i = 0$ (see previous paragraphs of this paper, for details). As a result, the conditions of homogeneity (6.1) and isotropy (6.2) take a simplified form: the conditions of homogeneity become

$${}^*\nabla_j F_i = 0\,, \qquad {}^*\nabla_j C_{ik} = 0\,, \qquad \frac{{}^*\partial p}{\partial x^i} = 0\,, \qquad (6.3)$$

while the conditions of isotropy become

$$F_i = 0\,, \qquad \Sigma_{ik} = 0\,. \qquad (6.4)$$

Let us calculate ${}^*\nabla_j C_{ik}$ and $\Sigma_{ik} = C_{ik} - \frac{1}{3}\,Ch_{ik}$ for the metrics (1.2) and (1.3), according to the formulae of C_{ik} obtained in §3 and §5, respectively. We obtain that ${}^*\nabla_j C_{ik} = 0$ and $\Sigma_{ik} = 0$ are satisfied in both cases, i.e. in both Schwarzschild bubbles and de Sitter bubbles.

However, $F_i \neq 0$ in both the metrics (1.2) and (1.3). This means that one of the conditions of isotropy (6.4), namely — $F_i = 0$, is violated in Schwarzschild bubbles and de Sitter bubbles.

Two conditions ${}^*\nabla_j F_i = 0$ and $\frac{{}^*\partial p}{\partial x^i} = 0$ of the conditions of homogeneity (6.3) remain for consideration.

First, we calculate ${}^*\nabla_j F_i = \frac{{}^*\partial F_i}{\partial x^j} - \Delta^m_{ji} F_m$ for the metric (1.2). We use the formula for F_1 (5.1), which is the solely non-zero component of the force. We obtain

$$ {}^*\nabla_1 F_1 = -\frac{c^2 r_g}{a^3}\,\frac{1}{\sqrt{1-\frac{r_g r^2}{a^3}}\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)} + $$

$$ + \frac{c^2 r_g^2}{a^6}\,\frac{r^2}{\left(1-\frac{r_g r^2}{a^3}\right)\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)^2} \neq 0\,, \qquad (6.5) $$

$$ {}^*\nabla_2 F_2 = \frac{{}^*\nabla_3 F_3}{\sin^2\theta} = -\frac{c^2 r_g}{a^3}\,\frac{r^2\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}} \neq 0\,. \qquad (6.6) $$

For the metric (1.3), we use the formula for F_1 (3.4). We obtain

$$ {}^*\nabla_1 F_1 = \frac{c^2 a^2}{(a^2-r^2)^2} \neq 0\,, \qquad {}^*\nabla_2 F_2 = \frac{{}^*\nabla_3 F_3}{\sin^2\theta} = \frac{c^2 r^2}{a^2} \neq 0 \qquad (6.7) $$

(these formulae can also be derived from the previous by substituting the condition $r_g = a$).

We see that the condition ${}^*\nabla_j F_i = 0$ is violated in both Schwarzschild bubbles and de Sitter bubbles.

Calculating $\frac{{}^*\partial p}{\partial x^i}$ for the metric (1.2), where the pressure p is expressed as (5.6), we obtain

$$ \frac{\partial p}{\partial r} = -\frac{2 r_g r}{a^3}\,\frac{\rho_0 c^2\sqrt{1-\frac{r_g}{a}}}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)^2} \neq 0\,, \qquad (6.8) $$

while for the metric (1.3), where $p = -\rho_0 c^2 = const$ (5.7), we have

$$ \frac{\partial p}{\partial r} = 0\,. \qquad (6.9) $$

In other words, the condition $\frac{{}^*\partial p}{\partial x^i} = 0$ is violated in Schwarzschild bubbles, but is satisfied in de Sitter bubbles.

Finally, we conclude:

> Space inside Schwarzschild bubbles and de Sitter bubbles is inhomogeneous and anisotropic due to the presence of the gravitational inertial force F_i. Also, the pressure p inside a Schwarzschild bubble is a function of distance, which generates an additional effect on the inhomogeneity of space.

At the same time, matter is homogeneously and isotropically distributed therein: this is incompressible liquid, which fills Schwarzschild bubbles, and physical vacuum (λ-field), which fills de Sitter bubbles. This is because the density of the liquid is $\rho_0 = const$ in Schwarzschild bubbles (despite $p \neq const$ therein), as well as $\rho_0 = const$ of physical vacuum (in the state of inflation, $p = -\rho_0 c^2$) in de Sitter bubbles. In brief, this situation can be resumed as follows:

> Despite the fact that space inside Schwarzschild bubbles is inhomogeneous and anisotropic, incompressible liquid is distributed homogeneously and isotropically therein. The same is true about de Sitter bubbles (filled with physical vacuum).

A short important note should be made concerning the gravitational inertial force F_i, which is the main factor of inhomogeneity and anisotropy of Schwarzschild bubbles and de Sitter bubbles.

Consider the space inside a de Sitter bubble. This is a de Sitter space, where the λ-term takes a particular value of $\lambda = \frac{3}{a^2} > 0$. In this case, de Sitter's metric takes the form (1.3) and, as was shown in §4, the λ-field has properties of ideal incompressible liquid in the state of inflation. We have already obtained F_1 for the metric (1.3). We calculate the regular (contravariant) vector F^1 of the gravitational inertial force from F_1 (3.4), by lifting the index with the contravariant chr.inv.-metric tensor h^{ik} (3.7). We obtain

$$F^1 = \frac{c^2 r}{a^2} = \frac{\lambda c^2 r}{3} \,. \tag{6.10}$$

Hubble's constant $H = (2.3 \pm 0.3) \times 10^{-18}$ sec^{-1} is expressed through the radius of the Universe $a = 1.3 \times 10^{28}$ cm as $H = \frac{c}{a}$. Taking this into account, we obtain

$$F^1 = H^2 r \,, \tag{6.11}$$

where Hubble's constant plays the rôle of a fundamental frequency. This formula meets the result recently obtained by Rabounski [11], according to which the Hubble redshift is due to the rotation of the isotropic space (home of photons) at the velocity of light. As was then shown [12], this effect is presented in any case, even if the non-isotropic space (home of solid bodies) does not rotate or deform.

Thus, according to the formula (6.11), the Hubble redshift has also been explained in the space inside de Sitter bubbles. This is despite that fact that the space does not expand or compress therein (it is free of deformation according to de Sitter's metric), i.e. the de Sitter bubble is a static cosmological model.

§7. Conclusion. In conclusion, we have arrived at Einstein's initial suggestion of de Sitter space as the basic cosmological model of our Universe (see page 5). Besides, it has been shown that this model satisfies the observed parameters of the Universe only in a particular case, where it is a collapsar (de Sitter bubble).

Among many advantages of the de Sitter bubble model, which have been elaborated upon in this paper, one of the most important is that the model allows us to calculate the characteristics of the Universe. This is in contrast to the Friedmann models, where, as is known, the parameter $R(t)$ is indefinite: this is an arbitrary function contained in the metric, so one should introduce it according to physical suggestions, which is not so satisfactory. In the de Sitter bubble model, the parameters of the Universe are unambiguously determined by the metric. All we need to do is substitute $a = r_g = \frac{2GM}{c^2}$ and the numerical values of the physical constants into the formulae obtained for the model.

For instance, let us substitute $a = 1.3\times10^{28}$ cm, which is the radius of our Universe. We obtain the following characteristics, which characterize the Universe as a de Sitter bubble

$$r_g = \frac{2GM}{c^2} = a = 1.3\times10^{28} \text{ cm}, \tag{7.1}$$

$$M = \frac{ac^2}{2G} = 8.8\times10^{55} \text{ gram}, \tag{7.2}$$

$$\rho_0 = \frac{3M}{4\pi a^3} = \frac{3c^2}{8\pi G a^2} = 9.5\times10^{-30} \text{ gram/cm}^3, \tag{7.3}$$

$$\lambda = \frac{3}{a^2} = \varkappa\rho_0 = 1.8\times10^{-56} \text{ cm}^{-2} \tag{7.4}$$

$$p = -\rho_0 c^2 = -\frac{\lambda c^2}{\varkappa} = -\frac{3c^2}{a^2\varkappa} = 8.6\times10^{-9} \text{ dynes/cm}^2. \tag{7.5}$$

These theoretical values correspond to those produced according to observational estimations. Therefore, the de Sitter bubble model suggested here is good enough to be a valid model of the Universe.

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Gravitational Waves and Gravitational Inertial Waves According to the General Theory of Relativity

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Abstract: This research concerns gravitational waves and gravitational inertial waves, considered as waves of the curvature of space (space-time). It was produced using the mathematical apparatus of chronometric invariants, which, being the projections of the four-dimensional quantities onto the line of time and the spatial section of an observer, are physically observable quantities. The wave functions (d'Alembertian) of the chronometrically invariant (physically observable) projections X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor are deduced. The conditions of the non-stationarity of the wave functions are taken into focus. It is shown that, even in the absence of the deformation of space ($D_{ik}=0$), the non-stationarity of the wave functions is possible. Four such cases were found, depending on the gravitational inertial force F_i and the rotation of space A_{ik}: 1) $F_i=0$, $A_{ik}=0$; 2) $F_i=0$, $A_{ik}\neq 0$; 3) $F_i\neq 0$, $A_{ik}=0$; 4) $F_i\neq 0$, $A_{ik}\neq 0$. It is shown that in the first case, where $F_i=0$ and $A_{ik}=0$, in emptiness, space is flat. If one of the quantities F_i and A_{ik} differs from zero, the metric remains stationary in emptiness and in the medium. If both F_i and A_{ik} are nonzero, the metric can be non-stationary in both emptiness and the medium, if the field F_i is vortical. The main conclusion is that it is not necessary that only the deformation of space be a source of gravitational waves and gravitational inertial waves. The waves can exist even in non-deforming spaces, if the gravitational inertial force F_i and the rotation of space A_{ik} differ from zero, and the field F_i is vortical.

Contents:

Preface 26

§1. Introduction 26

§2. The gravitational wave problem according to the classical theory of differential equations 29

§3. Generally covariant criteria for gravitational waves and their link to Petrov's classification 35

§4. The chronometrically invariant criterion for gravitational waves and its link to Petrov's classification 46

§5. Physical conditions of the existense of gravitational waves in non-empty spaces 57

§6. Chronometrically invariant representation of Petrov's classification for non-empty spaces 62

Preface. This is my BSc diploma study, which I produced during 1968 at the Sternberg Astronomical Institute of the Moscow State University, where I was a student in the years 1962–1969. I presented the study on January 27, 1969*. Six years later, on March 04, 1975, the Faculty of Theoretical Physics of the Patrice Lumumba University in Moscow considered this study (with minor changes) as a PhD thesis, and bestowed upon me a PhD degree†.

This study met much interest from the side of the local scientific community working in General Relativity and gravitation. This popularity, however, was very unfortunate to me: the person who had been formally nominated as my supervisor (despite the fact that I produced this research by my own solitary strength), had included my study, without any permission from my side, let alone a sense of dignity, as a substantial part of his book surveying the gravitational wave problem‡. This was a standard of poor behaviour in the formerly USSR, where a young researcher (especially among women, who are a significant minority among the scientists), even a highly potential one, was often treated with a deliberate dose of tyranny and neglect.

After four decades, I have decided to publish my first study in its original form, in accordance with my unpublished draft of 1968. This is because I think that the main research results (manifested in the resume outlined above) may still be of interest to the scientists working on the theory of gravitational waves.

September 16, 2010 *Larissa Borissova*

§1. Introduction. The gravitational wave problem remains unsolved until this day, in both the theoretical and experimental parts of it. The theoretical foundations for gravitational waves have arrive from the General Theory of Relativity. It is commonly accepted that the experimental registration of gravitational waves in the future will be one more direct verification of Einstein's field equations in particular, and Einstein's theory in general. Just after Albert Einstein introduced the General Theory of Relativity, Arthur Eddington considered a linearized

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‡Zakharov V. D. Gravitational Waves in Einstein's Theory of Gravitation. Translated by R. N. Sen, Halsted Press — John Wiley & Sons, Jerusalem — New York, 1973 (originally published in Russian by Nauka Publishers, Moscow, 1972).

form of Einstein's equations. He had found that the linearized equations have a non-stationary solution in emptiness. The discovered functions depend on the argument $ct + x^1$. Therefore, the non-stationary solution was interpreted as an elliptically polarized plane wave of the gravitational field (in other words, a *gravitational wave*) travelling in the direction x^1. Subsequently, Eddington suggested that the waves should transfer gravitational radiation, which was already predicted by Einstein. Commencing in the 1920's, this kind of solutions has been commonly assumed as a basis of the theory of gravitational waves. This is because the cosmic bodies which could theoretically be the sources of gravitational radiation are located very distant from the observer, thus the arriving gravitational wave can be assumed to be weak and plane.

Meanwhile, I am convinced that we should not limit ourselves to the single (simplest) metric of weak plane gravitational waves (I will refer to it as the *Einstein-Eddington metric*). We should consider the gravitational wave problem, including the Einstein-Eddington metric, from different viewpoints.

Apart from the Einstein-Eddington theory, outlined above, there are numerous other research directions, in which another determination has been applied to gravitational waves, thus introducing not only weak gravitational waves as in the Einstein-Eddington theory, but also strong gravitational waves, including also gravitational inertial waves.

Many problems can be met in this way. From a formal point of view, weak gravitational waves should serve as an approximation to strong gravitational waves. However, the problem concerning which definition should be applied to strong gravitational waves remains open until this day. Besides that, there is another serious problem: we still have not exact solution of the problem of the gravitational field energy. In other words, we still have not real energy-momentum tensor of the gravitational field in the theory, but only particular solutions of the problem: this is the energy-momentum pseudo-tensor of the field, in its different versions suggested by Einstein, Møller and Mitskievich, Stanyukovich, and others.

As follows from that has been said above, we still have not final clarity in the theoretical part of the gravitational wave problem. On the other hand, it is obvious that there are many non-stationary processes such as supernova explosions, binary star systems, and others, which, according to Einstein's theory, should produce gravitational radiation, thus filling space with gravitational waves travelling in all directions. In other words, the existence of gravitational waves is out of doubt. Hence, we should continue research in the theory of gravitational waves in looking for new approaches which could give a better chance for understanding the nature of the phenomenon. It is possible that one of the new approaches will give the energy-momentum tensor of the gravitational field, thus resolving the problem of the gravitational field energy, including the wave energy of the field.

Generally speaking, all theoretical studies of gravitational waves can be split into three main groups:

1) The first group consists of studies, which give a generally covariant definition for gravitational waves; the presence of such waves in space does not depend on the frame of reference of the observer. These are studies produced by Pirani [1], Lichnérowicz [2–4], Bel [5–8], Debever [9–11], Hély [12, 13], Trautman [14], Bondi [15], and others. I refer to it as the *generally covariant approach* to the gravitational wave problem.

2) The second group consists of studies, which give a chronometrically invariant definition for gravitational waves. This definition is invariant with respect to the transformations of time along the three-dimensional spatial section of the observer, and is based on the mathematical apparatus of chronometric invariants (physically observable quantities) introduced by Zelmanov [16,17]. Due to the common consideration of the fields of gravitation and rotation, which is specific to the mathematical apparatus, this definition is common to both gravitational waves (derived from masses) and gravitational inertial waves (derived from the fields of rotation) which thus are considered as two manifestations of the same phenomenon. These studies were started by Zelmanov himself (his results were surveyed by his student, Zakharov, in the publication of 1966 [18]), then continued in my early studies, and also in the present paper. I refer to it as the *chronometrically invariant approach.*

3) The third group joins studies around the search for such solutions of Einstein's equations, which, proceeding from physical considerations, could describe gravitational radiation. These are studies produced by Bondi [19], Einstein and Rosen [20, 21], Peres [22, 23], Takeno [24–26], Petrov [27], Kompaneetz [28], Robinson and Trautman [29, 30], and others. I refer to it as the *physical approach.*

Most criteria for gravitational waves were introduced proceeding from the properties of the Riemann-Christoffel curvature tensor. Therefore, it is commonly assumed that they are travelling waves of the curvature of space (space-time).

Besides that, the theory of gravitational waves is directly linked to the classification of spaces introduced by Alexei Petrov [27], which is known as *Petrov's classification.* This is a classification according to the algebraic structure of the Riemann-Christoffel curvature tensor. According to the classification, three main kinds of spaces (gravitational fields) exist. Petrov referred to them as *Einstein spaces*:

Einstein spaces of kind I. Fields of gravitation of kind I are derived from island-like distributions of masses. An example of such a field is that of a spherical mass, and is described by the Schwarzschild mass-point metric. Spaces containing such fields approach a flat space at an infinite distance from the gravitating island;

Einstein spaces of kinds II and III. Spaces filled with gravitational fields of kinds II and III cannot asymptotically approach a flat space even in the case where they are empty. Such spaces are curved themselves, independently of the presence of gravitating matter. They satisfy most of the invariant definitions given to gravitational waves [18, 29–32].

As is known (see Problem 1 to §102 *Gravitational Waves* in [33], and also page 41 herein), the metric of weak plane gravitational waves is one of the sub-kind N of kind II according to Petrov's classification.

Note that we mean herein the Riemannian (four-dimensional) curvature, whose formula contains the acceleration, rotation, and deformation of the observer's reference space. However, most analysis of the wave solutions to Einstein's equations has been limited to the idea that gravitational waves have a purely *deformational origin*, i.e. are waves of the deformation of space.

Thus, considering only all aforementioned physical factors of gravitational waves, we can arrive at understanding the true origin of the phenomenon. This is the main task of this study. We will do so by employing the mathematical methods of chronometric invariants.

§2. The gravitational wave problem according to the classical theory of differential equations. So, there are three main approaches to the gravitational wave problem according to the General Theory of Relativity: 1) the generally covariant criteria for gravitational waves, whose existence does not depend on our choice of the reference frame; 2) the chronometrically invariant approach, which gives definitions for both gravitational waves and gravitational inertial waves, determined in the real frame of reference connected to a real observer;

3) gravitational waves, defined on the basis of physical considerations. Before focusing on the approaches, I will consider the gravitational wave problem from the viewpoint of the classical theory of differential equations.

An exact theory of gravitational waves became possible after de Donder [34] and Lanczos [35] who proved that Einstein's equations are a system of partial differential equations of the hyperbolic kind. The classical theory of differential equations characterizes a wave by a *Hadamard break* [36] in the solutions of the wave equations in a hypersurface S along the wave front (named after Jacques Salomon Hadamard). The hypersurface wherein the field functions have a break is known as the wave front surface, and is a characteristic hypersurface of the field equations. Therefore, looking for characterstics of the hypersurface is one of the main tasks of the theory. The gravitational wave problem as a particular problem of the solutions to Einstein's equations is also linked to Cauchy's problem formulated for the system of quasi-linear partial differential equations of the hyperbolic kind. Solving this problem containing initial data depends on not only the class of smooth functions, but also on the initial shape of the hypersurface. Because Hadamard break plays a very important rôle in the further development of the generally covariant theory of gravitational waves, it is reasonable to say more on the topic.

At first, consider a scalar function ψ as an example. Let the function ψ be continuous in each of the neighbourhoods 1 and 2, obtained due to the hypersurface S which splits the given region of space (space-time). Let also the function ψ approach to the boundary numerical values ψ_1^0 and ψ_2^0, once x^α approaches to a point $P_0(x_0^\alpha)$ of the respective neighbourhoods 1 and 2 of the hypersurface. Given these assumptions, a break of the function ψ in the hypersurface S is the following function of the point P_0

$$[\psi]\,(P_0) = \psi_1^0 - \psi_2^0\,. \tag{2.1}$$

Let the function ψ be continuous everywhere near S, but several of their first partial derivatives $\frac{\partial\psi}{\partial x^\alpha}$ have finite breaks in S

$$[\psi] = 0\,, \qquad \left[\frac{\partial\psi}{\partial x^\alpha}\right] \neq 0\,. \tag{2.2}$$

Let the hypersurface S be determined by the equation $\varphi\,(x^\alpha\!=\!0)$. The normal vector $\frac{\partial\varphi}{\partial x^\alpha}$ of the hypersurface S is characterized by the relation

$$\frac{\partial\varphi}{\partial x^\alpha}\,dx^\alpha = 0\,, \qquad \alpha = 0,1,2,3, \tag{2.3}$$

if the increment dx^α lies in the hypersurface S. Hadamard [36] showed that, in this case, the break of the first derivative of the function is proportional to the derivative itself

$$\left[\frac{\partial\psi}{\partial x^\alpha}\right] = \chi\,\frac{\partial\varphi}{\partial x^\alpha}\,, \tag{2.4}$$

where χ is a coefficient of proportion. If the first derivatives are continuous, it is possible to show that the break of the second derivatives is expressed with the formula

$$\left[\frac{\partial^2\psi}{\partial x^\alpha\,\partial x^\beta}\right] = \chi\,\frac{\partial^2\varphi}{\partial x^\alpha\,\partial x^\beta}\,. \tag{2.5}$$

We are mostly interested in Cauchy's problem for the tensorial function $g_{\alpha\beta}$ obtained from Einstein's equations. In the case of Einstein's equations which determine an empty field of gravitation

$$R_{\alpha\beta} = 0\,, \tag{2.6}$$

where $R_{\alpha\beta}$ is Ricci's tensor, Cauchy's problem is formulated as follows:

Cauchy's problem. Consider an initial hypersurface S described by the equation

$$\varphi\,(x^\alpha) = 0\,. \tag{2.7}$$

Let functions $g_{\alpha\beta}\,(x^\sigma)$ and their first derivatives $\frac{\partial g_{\alpha\beta}(x^\sigma)}{\partial x^\rho}$ are present on the hypersurface. It is required to find these functions outside the hypersurface S given that they and their first derivatives meet the respective functions on the hypersurface S, and that all functions $g_{\alpha\beta}$ satisfy Einstein's equations in emptiness.

A Hadamard break of a tensorial function $g_{\mu\nu}$ is determined according to the equation

$$\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha\,\partial x^\beta}\right] = a_{\mu\nu}\,\frac{\partial^2\varphi}{\partial x^\alpha\,\partial x^\beta}\,, \tag{2.8}$$

where $a_{\mu\nu}$ are *coefficients of the breaking* [14, 37]. The studies [14, 37] manifest that, given all second derivatives of the function $g_{\mu\nu}$, only those by x^0 can experience some breaking in the hypersurface S

$$\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^0\,\partial x^0}\right] = a_{\mu\nu}\,. \tag{2.9}$$

Concerning Einstein's equations, this problem seems more particular. As is known, Einstein's equations do not contain the second derivatives of $g_{0\alpha}$ with respect to $x^0 = ct$. It is important to know that, given all second derivatives of $g_{\mu\nu}$ which are included into Einstein's equations, only the second derivatives of the three-dimensional components g_{ij} by x^0 (where $i, j = 1, 2, 3$) can experience a break in the hyperspace S. André Lichnérowicz [38] had showed that Einstein's equations in emptiness, considered under the following condition $g^{00} \neq 0$, can have a solution which has not a Hadamard break in S. This coincides with the case where the second derivatives of g_{ij} by x^0 are unambiguously determined in common with the Cauchy initial data. If, however, $g^{00} = 0$ in the neighbourhood of S, the derivatives and, hence, the respective components R_{0i0j} of the Riemann-Christoffel curvature tensor cannot be unambiguously determined by the Cauchy initial data and Einstein's equations, thus the second derivatives of g_{ij} with respect to x^0 experience a Hadamard breaking in the hypersurface S. This is known as a *Hadamard weak break of the 1st kind.*

The condition $g^{00} = 0$, which determines the Hadamard break of the Riemann-Christoffel curvature tensor in the initial hypersurface, can be re-formulated in the generally covariant form

$$g^{\alpha\beta}\,\frac{\partial^2\varphi}{\partial x^\alpha\,\partial x^\beta} = 0\,, \tag{2.10}$$

which is the same as the eikonal equation (equation of the wave phase) known in geometrical optics. This is a necessary and sufficient condition of the isotropy of the hypersurface S. Hence, the break of the Riemann-Christoffel curvature tensor, which is the condition of that the gravitational field in an empty space is a wave, is possible only if the initial hypersurface is isotropic.

Lichnérowicz [39] had proven the following theorem (I refer to it as *Lichnérowicz' theorem*):

Lichnérowicz' theorem. A Hadamard break of the curvature tensor $R_{\alpha\beta\gamma\delta}$ in an empty space is possible only in the characteristic hypersurface S of Einstein's equations in emptiness, which is determined by the eikonal equation.

A *characteristic hypersurface* is thus such that satisfies the eikonal equation. An enveloping arc of the hyperplanes, which are tangential to all hypersurfaces which are conceived at the given point, is known as a *characteristic cone* [40].

Because the characteristic hypersurface of Einstein's equations in emptiness is isotropic (the interval of length is zero therein), the characteristic cone of Einstein's equations meets the light cone in an empty space [38]. *Bicharacteristics* of Einstein's equations, known also as *rays*, meet the lines of the current of a vectorial field l^{α}, which is orthogonal to the characteristic hypersurface S

$$l^{\alpha} = g^{\alpha\beta}\,\frac{\partial\varphi}{\partial x^{\beta}}\,, \tag{2.11}$$

and are characterized by the equation

$$\frac{dx^{\alpha}}{d\tau} = g^{\alpha\beta}\,\frac{\partial\varphi}{\partial x^{\beta}}\,, \tag{2.12}$$

where τ is a nonzero parameter taken along the ray. Lichnérowicz [38] also showed that the functions of x^{α} are geodesics of a Riemannian space, whose metric is $g_{\alpha\beta}$.

The theory of partial differential equations says that the bicharacteristics (rays) belong to the charactersic hypersurface, hence the lines oriented tangentially to them are elements of the characteristic cone, which, in this case, meets the light cone [38]. The following conclusion follows herefrom:

> The travelling rays of gravitational waves are isotropic geodesic lines, as well as the travelling rays of light.

Proceeding from this analogy, Lichnérowicz [38] considered Cauchy's problem for Maxwell's equations in a Riemannian space V_4. He had proved the following theorem (I refer to it as *Lichnérowicz' theorem on characteristic manifolds*):

Theorem on characteristic manifolds. The characteristic manifolds of Einstein's equations and Maxwell's equations meet each other in a Riemannian space V_4, and are determined by the solution of the eikonal equation of these fields.

Analysis of this theorem, while taking into account that has been said on the rays of the travel of gravitational waves, necessarily leads to the obvious conclusion:

> The bicharacteristics of Einstein's equations (the *rays of gravitational waves*) coincide with the bicharacteristics of Maxwell's equations (the *rays of electromagnetic waves*). Thus, proceeding from the classical theory of differential equations, gravitational waves and electromagnetic waves travel at the velocity of light, along the same isotropic geodesics.

In brief, the main results obtained due to the classical theory of differential equations are such that the characteristic manifold of Einstein's equations is a hypersurface, wherein the Riemann-Christoffel curvature tensor has a Hadamard break. Therefore, this hypersurface is the front of a gravitational wave. The bicharacteristics of Einstein's equations are trajectories of an isotropic vector, which is orthogonal to the wave front, thus this is a wave vector. Because the characteristics of the characteristic manifold are generally covariant quantities, the hypersurface of Einstein's equations can be considered as an invariantly determined front of a gravitational wave, while the bicharacteristics of Einstein's equations — as invariantly determined rays. The front of an electromagnetic wave in a Riemannian space V_4 is determined by the characteristic hypersurface of Maxwell's equations. According to Lichnérowicz' theorem on the characteristic manifolds, the front of an electromagnetic wave coincides with the front of a gravitational wave, while the electromagnetic rays (bicharacteristics of Maxwell's equations) coincide with the gravitational rays (bicharacteristics of Einstein's equations).

Despite having a general method determining gravitational waves as kinds of Einstein's equations in emptiness, or as kinds of the Einstein-Maxwell equations in a space filled with both gravitational and electromagnetic fields, we cannot obtain exact solutions of the system of Einstein's equations (or the Einstein-Maxwell equations), because we meet the following difficulties:

1) Einstein's equations have a complicate non-linear structure. They have not universal boundary conditions;
2) We have not an universal form of d'Alembert's operator, which could be explicitly expressed from Einstein's equations. The core of this problem is that the unknown variables of Einstein's equations are the components of the fundamental metric tensor $g_{\alpha\beta}$, which conserves in the generally covariant meaning: it satisfies the generally covariant conservation law, thus $\nabla_\sigma g_{\alpha\beta} = 0$ (here ∇_σ is the symbol of absolute differentiation). Therefore, the generally covariant d'Alembertian of the fundamental metric tensor is zero: $\Box g_{\alpha\beta} \equiv g^{\rho\sigma}\nabla_\rho\nabla_\sigma g_{\alpha\beta} \equiv 0$.

Einstein's theory interprets gravitational fields as distortions of space (space-time). Therefore, it is a naturally valid idea to connect gravitational waves to the properties of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$. The four-dimensional pseudo-Riemannian space, which is the basic space-time of General Relativity, is characterized by the curvature tensor: if the tensor is zero in a region, gravitational fields are absent therein. The Riemann-Christoffel curvature tensor is not a direct part of Einstein's equations. Only its contacted forms, namely — Ricci's tensor and scalar, form the basis of the equations. Therefore, other methods should be applied in order to study its structure. In particular, we can impose some conditions (criteria) on the tensor, which could allow to consider the curvature field as a gravitational wave. In this direction, an emergent goal in the theory of gravitational waves was included due to studies of the algebraic properties of the Riemann-Christoffel curvature tensor produced by Petrov [27]. His classification of the curvature tensor according to its algebraic structure allowed him to determine several kinds of the solutions of Einstein's equations as gravitational wave fields.

We will consider the invariant criteria for gravitational waves, and also Petrov's results related to the algebraic structure of the curvature tensor, in the next paragraphs §3 and §4.

§3. Generally covariant criteria for gravitational waves and their link to Petrov's classification. As was mentioned in the end of §1, most analysis of the wave solutions to Einstein's equations was limited by an idea that they are only due to the factor of the deformation of space, thus gravitational waves are waves of the deformation of space. Here the next question arises. How well is this statement justified? General covariant criteria for the wave solutions to Einstein's equations will be our task in this paragraph.

Einstein's equations (gravitational field equations) have the form

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,, \tag{3.1}$$

where $R_{\alpha\beta} = R^{\sigma\cdots}_{\cdot\alpha\sigma\beta}$ is Ricci's tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, $\varkappa = \frac{8\pi G}{c^2}$ is Einstein's gravitational constant, G is Gauss' constant of gravitation, λ is the cosmological term.

When studying gravitational waves, most scientists assume $\lambda = 0$ thus considering a particular case of Einstein's equations, which is

$$R_{\alpha\beta} = \kappa g_{\alpha\beta}\,. \tag{3.2}$$

This is a case of spaces known, after Petrov [27], as *Einstein spaces.* They can be either empty ($\kappa = 0$) or filled with homogeneously distributed matter (in this case, $R_{\alpha\beta} \sim \varkappa T_{\alpha\beta}$). If $\kappa = 0$ in an Einstein space, there is not distributed matter. If there is not islands of mass as well, such an empty space can also be curved: in this case, it is related to kinds II and III according to Petrov's classification (see page 29).

As was mentioned in §2, according to the classical theory of differential equations, gravitational wave fields are determined by solutions of Einstein's equations, taken with the initial conditions of a characteristic hypersurface of the equations. A gravitational wave is a Hadamard break in the initial characteristic hypersurface of the equations; this surface is the front of a gravitational wave. Let us re-write the formula of a Hadamard break of a tensorial function $g_{\mu\nu}$ in a Riemannian space, (2.8), as

$$\left[\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\right] = a_{\mu\nu}\, l_\alpha l_\beta\,, \qquad l_\alpha \equiv \frac{\partial\varphi}{\partial x^\alpha}\,. \tag{3.3}$$

According to Lichnérowicz [2–4], who followed with Hadamard's studies, a Hadamard break of the second derivatives of $g_{\mu\nu}$ can be in a characteristic hypersurface of Einstein's equation only due to a Hadamard break in the field of the Riemann-Christoffel curvature tensor, i.e. due to $[R_{\alpha\beta\gamma\delta}]$, which satisfies the equations (see [4])

$$l_\lambda\,[R_{\mu\alpha\beta\nu}] + l_\alpha\,[R_{\mu\beta\lambda\nu}] + l_\beta\,[R_{\mu\lambda\alpha\nu}] = 0\,. \tag{3.4}$$

Proceeding from this condition realized in a characteristic hypersurface of Einstein's equations, and also because the break $[R_{\alpha\beta\gamma\delta}]$, located at the front of a gravitational wave, is proportional to the curvature tensor $R_{\alpha\beta\gamma\delta}$ itself (see §2 herein for detail), Lichnérowicz was able to formulate his generally covariant criterion for gravitational waves [2–4]:

Lichnérowicz' criterion. The Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta} \neq 0$ determines the state of "pure gravitational radiation", only if there is a vector l^α, which is orthogonal to the characteristic surface of Einstein's equations, is isotropic ($l_\alpha l^\alpha = 0$), and satisfies the equations

$$\left.\begin{array}{l} l^\mu R_{\mu\alpha\beta\nu} = 0 \\ l_\lambda R_{\mu\alpha\beta\nu} + l_\alpha R_{\mu\beta\lambda\nu} + l_\beta R_{\mu\lambda\alpha\nu} = 0 \end{array}\right\}. \tag{3.5}$$

If $R_{\alpha\beta} = 0$ (in an empty space, which is free of distributed matter of any kind), the equations determine the field of "pure gravitational radiation".

There is also another generally covariant criterion for gravitational waves, formulated by Zelmanov* [18]. It is indirectly connected to Lichnérowicz' criterion. Zelmanov's criterion proceeds from the d'Alembert generally covariant operator

$$\Box^\sigma_\sigma \equiv g^{\rho\sigma}\,\nabla_\rho \nabla_\sigma\,, \tag{3.6}$$

and is formulated as follows:

*This criterion was introduced by Abraham Zelmanov in the early 1960's, and was presented to a close circle of his associates. It was first published in 1966, in the survey on the gravitational wave problem [18] authored by Zakharov, who was a student of Zelmanov. Zakharov referred to Zelmanov in the publication.

Zelmanov's criterion. A space satisfies the state of gravitational radiation, only if the Riemann-Christoffel curvature tensor a) does not conserve ($\nabla_\sigma R_{\mu\alpha\beta\nu} \neq 0$), and b) satisfies the generally covariant condition

$$\Box^\sigma_\sigma R_{\mu\alpha\beta\nu} = 0\,. \tag{3.7}$$

Any empty space, satisfying Zelmanov's criterion, satisfies Lichnérowicz' criterion as well. And vice versa: any empty space, which satisfies Lichnérowicz' criterion (excluding constant curvature spaces, where $\nabla_\sigma R_{\mu\alpha\beta\nu} = 0$), also satisfies Zelmanov's criterion.

There are also numerous other generally covariant criteria for gravitational waves, introduced by Bel, Pirani, Debever, Maldybaeva and others. Each of the criteria has its own advantages and drawbacks, therefore none of the criteria can be considered as the final solution of the gravitational wave problem. Therefore, it would be a good idea to consider those characteristics of gravitational wave fields, which are common to most of the criteria. Such an integrating factor is Petrov's classification according to the algebraic structure of the Riemann-Christoffel curvature tensor [27]. This is a classification of spaces, which satisfy the particular Einstein equations (3.2) and are known as Einstein spaces. Thus, gravitational fields, which satisfy (3.2), can also be classified in this way.

As is known, the Riemann-Christoffel curvature tensor satisfies the following identities

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}\,, \qquad R_{\alpha[\beta\gamma\delta]} = 0\,. \tag{3.8}$$

Because of these identities, the curvature tensor is related to tensors of a special family, known as *bitensors*. They satisfy two conditions:

1) Their covariant and contravariant valencies are even;
2) Both covariant and contravariant indices of the tensors are split into pairs, and inside each pair the tensor $R_{\alpha\beta\gamma\delta}$ is antisymmetric.

A set of tensor fields located in an n-dimensional Riemannian space is known as a *bivector set*, and its representation at a point is known as a *local bivector set*. Every antisymmetric pair of indices $\alpha\beta$ is denoted by a common index a, and the number of common indices is $N = \frac{n(n-1)}{2}$. It is obvious that if $n = 4$ we have $N = 6$. Hence a bitensor $R_{\alpha\beta\gamma\delta} \rightarrow R_{ab}$, located in a four-dimensional Riemannian space, maps itself into a six-dimensional bivector space. It can be metrized by introducing the specific metric tensor

$$g_{ab} \rightarrow g_{\alpha\beta\gamma\delta} \equiv g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\,. \tag{3.9}$$

The metric tensor g_{ab} (where $a, b = 1, 2, \dots N$) is symmetric and non-degenerate. If the metric is given for the sign-alternating $g_{\alpha\beta}$, it is sign-alternating as well, having a respective signature. So, for Minkowski's signature $(+---)$ of $g_{\alpha\beta}$, the signature of g_{ab} is $(+++---)$.

Mapping the curvature tensor $R_{\alpha\beta\gamma\delta}$ onto the metric bivector space V_N, we obtain a symmetric tensor R_{ab} (where $a, b = 1, 2, \dots N$) which can be associated with a lambda-matrix

$$\| R_{ab} - \Lambda g_{ab} \| . \tag{3.10}$$

Solving the classical problem of linear algebra (reducing a lambda-matrix to its canonical form along a real distance), we can find a classification for V_n under a given n. A specific kind of spaces, which are Einstein spaces we are considering, is set up by a characteristic of the respective lambda-matrix. This kind remains unchanged in the area, where this characteristic remains unchanged.

Bases of the elementary divisors of the lambda-matrix for any V_n have an ordinary geometric meaning as *stationary curvatures.* Naturally, the Riemannian curvature K of V_n in a two-dimensional direction is determined by an ordinary (single-sheet) bivector $V^{\alpha\beta} = V^{\alpha}_{(1)} V^{\beta}_{(2)}$ as

$$K = \frac{R_{\alpha\beta\gamma\delta}\, V^{\alpha\beta} V^{\gamma\delta}}{g_{\alpha\beta\gamma\delta}\, V^{\alpha\beta} V^{\gamma\delta}} . \tag{3.11}$$

If $V^{\alpha\beta}$ is non-ordinary, the invariant K is known as the *bivector curvature in the direction of the vector.* Mapping K onto the bivector space, we obtain

$$K = \frac{R_{ab} V^a V^b}{g_{ab} V^a V^b} , \qquad a, b = 1, 2, \dots N. \tag{3.12}$$

The ultimate numerical values of K are known as *stationary curvatures at a given point*, while the vectors V^a corresponding to them are known as *stationary non-simple bivectors.* In this case

$$V^{\alpha\beta} = V^{\alpha}_{(1)} V^{\beta}_{(2)} , \tag{3.13}$$

so the stationary curvature is the same as the Riemannian curvature in the given two-dimensional direction.

Finding the ultimate numerical values of K is the same as finding those vectors V^a, where K takes the ultimate numerical values. This is the same as finding *undoubtedly stationary directions.* The necessary and sufficient condition of a stationary state of V^a is

$$\frac{\partial}{\partial V^a} K = 0 . \tag{3.14}$$

The problem of finding the stationary curvatures for Einstein spaces had been solved by Petrov [27]. If the space metric is sign-alternating, the stationary curvatures are complex as well as the stationary bivectors relating to them in the space V_n. For Einstein spaces of four dimensions with Minkowski's signature, Petrov had formulated a theorem:

Petrov's theorem. Given an ortho-frame $g_{\alpha\beta} = \{+1, -1, -1, -1\}$, there is a symmetric paired matrix $\|R_{ab}\|$

$$\|R_{ab}\| = \left\| \begin{array}{c|c} M & N \\ \hline N & -M \end{array} \right\|, \tag{3.15}$$

where M and N are two symmetric square matrices of the 3rd order, whose components satisfy the relationships

$$m_{11} + m_{22} + m_{33} = -\kappa\,, \qquad n_{11} + n_{22} + n_{33} = 0\,. \tag{3.16}$$

After transformations, the lambda-matrix $\|R_{ab} - \Lambda g_{ab}\|$, where $g_{ab} = \{+1, +1, +1, -1, -1, -1\}$, takes the form

$$\|R_{ab} - \Lambda g_{ab}\| = \left\| \begin{array}{c|c} M + iN + \Lambda\varepsilon & 0 \\ \hline 0 & M - iN + \Lambda\varepsilon \end{array} \right\| \equiv$$

$$\equiv \left\| \begin{array}{cc} Q(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda) \end{array} \right\|, \tag{3.17}$$

where $Q(\Lambda)$ and $\bar{Q}(\Lambda)$ are three-dimensional matrices, whose elements are complex conjugates, and ε is the three-dimensional unit matrix.

The matrix $Q(\Lambda)$ can have only one of the following three kinds of characteristics: I) [111]; II) [21]; III) [3].

As a matter of fact that the initial lambda-matrix can have only one characteristic drawn from: I) $[111, \overline{111}]$; II) $[21, \overline{21}]$; III) $[3, 3]$.

The numbers in brackets means the multiplicity of roots of the characteristic equation $\det \|R_{ab} - \Lambda g_{ab}\| = 0$ (see Chapter 2 in [27]). Consider a 6×6 matrix g_{ab}. Construct the characteristic equation for it. This is a 6th order equation: it has 6 roots and, as Petrov showed, the ultimate number of different roots is 3 as for a 3×3 matrix (also several of these 3 pairs of roots can be complex conjugates). Obtain the roots, then compare the obtained pairs of solutions. If all 3 pairs of roots differ from each other, this is kind [111]. If two of them are the same, this is kind [21]. If all 3 pairs of roots are the same, this is kind [3].

The bar in the second half of a characteristic means an imaginary part of the complex conjugates. There is not bar in kind $[3, 3]$, because the elementary divisors are always real therein.

Taking a lambda-matrix of each of three possible kinds, Petrov [27] had deduced the canonical form of the matrix $\|R_{ab}\|$ in a non-holonomic ortho-frame

$$\left.\begin{array}{c}\underline{\text{Kind I}}\\[4pt] \|R_{ab}\| = \begin{Vmatrix} M & N \\ N & -M \end{Vmatrix}, \\[8pt] M = \begin{Vmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{Vmatrix}, \quad N = \begin{Vmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{Vmatrix} \end{array}\right\}, \tag{3.18}$$

where $\sum_{i=1}^{3}\alpha_i = -\kappa$ and $\sum_{i=1}^{3}\beta_i = 0$ (so, here are 4 independent parameters, determining the space structure by an invariant form),

$$\left.\begin{array}{c}\underline{\text{Kind II}}\\[4pt] \|R_{ab}\| = \begin{Vmatrix} M & N \\ N & -M \end{Vmatrix}, \\[8pt] M = \begin{Vmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2+1 & 0 \\ 0 & 0 & \alpha_2-1 \end{Vmatrix}, \quad N = \begin{Vmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{Vmatrix} \end{array}\right\}, \tag{3.19}$$

where $\alpha_1 + 2\alpha_2 = -\kappa$ and $\beta_1 + 2\beta_2 = 0$ (here are 2 independent parameters determining the space structure by an invariant form),

$$\left.\begin{array}{c}\underline{\text{Kind III}}\\[4pt] \|R_{ab}\| = \begin{Vmatrix} M & N \\ N & -M \end{Vmatrix}, \\[8pt] M = \begin{Vmatrix} -\frac{\kappa}{3} & 1 & 0 \\ 1 & -\frac{\kappa}{3} & 0 \\ 0 & 0 & -\frac{\kappa}{3} \end{Vmatrix}, \quad N = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix} \end{array}\right\}, \tag{3.20}$$

thus no independent parameters determining the space structure by an invariant form exist in this case.

Thus Petrov has successfully resolved the problem of reducing a lambda-matrix to its canonical form along a real path in a Riemannian space with a sign-alternating metric. Despite the fact that his solution is obtained only at a given point, the obtained classification is invariant because the results are applicable to any point in the space.

Stationary curvatures take the form

$$\Lambda_i = \alpha_i + i\beta_i \tag{3.21}$$

in spaces of kind III, where they take real values ($\Lambda_1 = \Lambda_2 = \Lambda_3 = -\frac{\kappa}{3}$).

Numerical values of some stationary curvatures in spaces (gravitational fields) of kinds I and II can coincide with each other. If they are the same, we have sub-kinds of the spaces (fields). Kind I has 3 sub-kinds: I ($\Lambda_1 \neq \Lambda_2 \neq \Lambda_3$); D ($\Lambda_2 = \Lambda_3$); O ($\Lambda_1 = \Lambda_2 = \Lambda_3$). If the space is empty ($\kappa = 0$), the sub-kind O of kind I gives a flat space. Kind II has 2 sub-kinds: II ($\Lambda_1 \neq \Lambda_2$, $\Lambda_2 = \Lambda_3$) and N ($\Lambda_1 = \Lambda_2$). Kinds I and II are the basic kinds of Petrov's classifications.

In empty spaces (empty gravitational fields) the stationary curvatures are $\Lambda = 0$, so empty spaces (fields) are degenerate.

Studies of the algebraic structure of the Riemann-Christoffel curvature tensor for known solutions of Einstein's equations showed that most of the solutions are related to kind I. The curvature decreases with distance from a gravitating mass. In the extreme case, where the distance becomes infinite, the space approaches a flat space. As was shown in my early (unpublished) study, reported to Zelmanov when I was a student, the Schwarzschild mass-point solution, which represents a spherically symmetric gravitational field derived from an island of mass located in emptiness, is classified as the sub-kind D of kind I.

General covariant criteria for gravitational waves are linked to the algebraic structure of the curvature tensor, and thus should be associated with the aforementioned types of Einstein spaces. Most gravitational wave solutions of Einstein's equations are attributed to the sub-kind N of kind I. Several gravitational wave solutions are attributed to kinds II and III. Note, spaces of kinds II and III cannot approach a flat space, because the components of the curvature tensor matrix $\|R_{ab}\|$ contain $+1$ and -1. This makes the approach of the curvature tensor to zero impossible, and thus excludes approaching a flat space at infinity. Therefore, gravitational waves (waves of the curvature) are present everywhere in spaces of kinds II and III. Pirani [1] holds that gravitational waves are solutions to the gravitational field equations in spaces of the sub-kind N of kind II, or of kind III. The following solutions are classified to the sub-kind N of kind II: Peres' solution [22, 23] which describes plane gravitational waves

$$ds^2 = (dx^0)^2 - 2\alpha(dx^0 + dx^3)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \quad (3.22)$$

Takeno's solution [24–26]

$$\begin{aligned} ds^2 = (\gamma + \rho)\,(dx^0)^2 - 2\rho\, dx^0 dx^3 - \\ - \alpha(dx^1)^2 - 2\delta\, dx^1 dx^2 - \beta(dx^2)^2 + (\rho - \gamma)\,(dx^3)^2, \end{aligned} \quad (3.23)$$

where $\alpha = \alpha(x^1 - x^0)$, while γ, ρ, β, δ are functions of $(x^3 - x^0)$, and also Petrov's solution [27], which was represented by Bondi, Pirani, and Robertson in another coordinate system [15] as

$$ds^2 = (dx^0)^2 - (dx^1)^2 + \alpha(dx^2)^2 + 2\beta\, dx^2 dx^3 + \gamma(dx^3)^2, \qquad (3.24)$$

where α, β, γ are functions of $(x^1 + x^0)$.

A detailed survey of the relations between the generally covariant criteria for gravitational waves and Petrov's classification was presented in the publication [18]. Among the other issues, two following theorems were discussed therein:

Theorem. In order that a space satisfies the state of "pure gravitational radiation" (in the Lichnérowicz sense), it is a necessary and sufficient condition that the space is an Einstein space of the sub-kind N of kind II according to Petrov's classification, thus characterized by zero curvature matrix $\|R_{ab}\|$ in the bivector space.

Theorem. An Einstein space satisfying Zelmanov's criterion can only be an empty space $(\kappa = 0)$ of the sub-kind N of kind II. And vice versa, any empty space V_4 of the sub-kind N satisfies Zelmanov's criterion as well. This is true excluding symmetric spaces* of this kind; symmetric spaces of this kind have the metric

$$ds^2 = 2\, dx^0 dx^1 - \operatorname{sh}^2 dx^0 (dx^2)^2 - \sin^2 dx^0 (dx^3)^2. \qquad (3.25)$$

Proceeding from the theorems, we immediately arrive at a relation between Zelmanov's criterion for gravitational wave fields in emptiness and Lichnérowicz' criterion for "pure gravitational radiation":

Theorem. Any empty space V_4, satisfying Zelmanov's criterion for gravitational wave fields located in empty spaces, also satisfies Lichnérowicz' criterion for "pure gravitational radiation". And vice versa, any empty space V_n, satisfying Lichnérowicz' criterion (excluding the case of symmetric spaces), satisfies Zelmanov's criterion as well.

How are these criteria related to each other in a general case? This problem is still open for discussion.

In [18] it was shown that all known solutions to Einstein's equations in emptiness, which satisfy Zelmanov's criterion and Lichnérowicz' criterion, can be obtained as particular cases of a generalized metric whose space permits a vector field l^α, which conserves in the space and thus

*A space is referred to as *symmetric*, if its curvature tensor $R_{\alpha\beta\gamma\delta}$ conserves and thus satisfies the conservation condition $\nabla_\sigma R_{\alpha\beta\gamma\delta} = 0$.

satisfies the conservation law

$$\nabla_\sigma\, l^\sigma = 0\,. \tag{3.26}$$

It is obvious that this condition leads to Lichnérowicz' condition (3.5), hence this empty space is classified as the sub-kind N of kind II, and, also, the vector l^α playing the rôle of a gravitational wave vector, is unique and isotropic $l_\alpha l^\alpha = 0$. According to Eisenhart's theorem [41], a Riemannian space V_4 containing a unique isotropic vector l^α (in other words, an absolute parallel vector field), has the metric

$$\begin{aligned} ds^2 = \varepsilon\,(dx^0)^2 + 2\,dx^0 dx^1 + 2\varphi\, dx^0 dx^2 + \\ + 2\psi\, dx^0 dx^3 + \alpha(dx^2)^2 + 2\gamma\, dx^2 dx^3 + \beta(dx^3)^2, \end{aligned} \tag{3.27}$$

where ε, φ, ψ, α, β, γ are functions of x^0, x^2, x^3, and $l^\alpha = \delta^\alpha_1$. This metric satisfies the particular form (3.2) of Einstein's equations. So this is an exact solution to Einstein's equations in emptiness or vacuum, and satisfies Zelmanov's criterion and Lichnérowicz' criterion for gravitational waves. This solution generalizes those solutions suggested by Takeno, Peres, Bondi, Petrov and others, which satisfy the Zelmanov and Lichnérowicz criteria.

The metric (3.27), taken under additional conditions suggested by Bondi [18], satisfies Einstein's equations in their general form (3.1) in the case where $\lambda = 0$ and the energy-momentum tensor $T_{\alpha\beta}$ describes an isotropic electromagnetic field. Given an isotropic electromagnetic field, Maxwell's tensor $F_{\mu\nu}$ of the field satisfies the conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.28}$$

where $F^{*\mu\nu} = \frac{1}{2}\,\eta^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is a pseudotensor dual to Maxwell's tensor, while $\eta^{\mu\nu\rho\sigma}$ is the completely antisymmetric discriminant tensor (it makes pseudotensors out of tensors). Direct substitution shows that this metric satisfies the following requirements: the Riner-Wheeler condition discussed by Peres [42]

$$R = 0\,, \qquad R_{\alpha\rho}R^{\rho\beta} = \frac{1}{4}\,\delta^\beta_\alpha\,(R_{\rho\sigma}R^{\rho\sigma}) = 0\,, \tag{3.29}$$

where $\delta^\alpha_\beta = g^\alpha_\beta$, and the Nordtvedt-Pagels condition [43]

$$\eta_{\mu\varepsilon\gamma\sigma}\left(R^{\delta\gamma,\sigma}R^{\varepsilon\tau} - R^{\delta\varepsilon,\sigma}R^{\gamma\tau}\right) = 0\,, \tag{3.30}$$

where $R^{\delta\gamma,\sigma} = g^{\sigma\mu}\nabla_\mu R^{\delta\gamma}$.

We have an interest in isotropic electromagnetic fields because an observer, who accompanies such a field, should be moving at the velocity of light [1, 4]. Hence, isotropic electromagnetic fields can be interpreted as fields of electromagnetic radiation without sources. On the other hand, according to Eisenhart's theorem [41], a Riemannian space V_4 having the metric (3.27) permits an absolute parallel vector field l^{α}. Therefore, we conclude that the vector l^{α} considered in this case satisfies Lichnérowicz' criterion for "pure gravitational radiation".

Thus the metric (3.27), satisfying the conditions

$$\left.\begin{aligned} & R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, T_{\alpha\beta} \\ & T_{\alpha\beta} = \frac{1}{4}\, F_{\rho\sigma} F^{\rho\sigma} g_{\alpha\beta} - F_{\alpha\sigma} F_{\beta\cdot}^{\cdot\sigma} \\ & F_{\alpha\beta} F^{\alpha\beta} = 0\,, \qquad F_{\alpha\beta} F^{*\alpha\beta} = 0 \end{aligned}\right\} \tag{3.31}$$

and taken under the additional condition suggested by Bondi [18]

$$R_{2323} = R_{0232} = R_{0323} = 0\,, \tag{3.32}$$

is an exact solution to Einstein's equations, which describes both gravitational waves and electromagnetic waves without sources. This solution does not satisfy Zelmanov's criterion in a general case, but satisfies it in particular cases where $T_{\alpha\beta} \neq 0$, and also $R_{\alpha\beta} \neq 0$.

A recursion curvature space is a Riemannian space, which has a curvature satisfying the relationship

$$\nabla_{\sigma} R_{\alpha\beta\gamma\delta} = l_{\sigma} R_{\alpha\beta\gamma\delta}\,. \tag{3.33}$$

Due to Bianchi's identity, such a space satisfies

$$l_{\sigma} R_{\alpha\beta\gamma\delta} + l_{\alpha} R_{\beta\sigma\gamma\delta} + l_{\beta} R_{\sigma\alpha\gamma\delta} = 0\,. \tag{3.34}$$

A common classification for recursion curvature spaces had been suggested by Walker [44]. His classification was then applied to the four-dimensional pseudo-Riemannian space (the basic space-time of General Relativity). Concerning the class of prime recursion spaces*, we are particularly interested in two metrics, which are

$$ds^2 = \psi\,(x^0, x^2)(dx^0)^2 + 2\,dx^0 dx^1 - (dx^2)^2 - (dx^3)^2, \tag{3.35}$$

$$ds^2 = 2\,dx^0 dx^1 + \psi\,(x^1, x^2)(dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{3.36}$$

*A recursion curvature space is prime or simple, if it contains $n-2$ parallel vector fields (isotropic and non-isotropic). Here n is the dimension of the space.

where $\psi > 0$. In these metrics, only one component of Ricci's tensor is nonzero. It is $R_{00} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^0 \partial x^0}$ in (3.35), and $R_{11} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^1 \partial x^1}$ in (3.36). Einstein spaces with these metrics can only be empty ($\kappa = 0$) and flat ($R_{\alpha\beta\gamma\delta} = 0$). This can be proved by checking that both metrics satisfy the Riner-Wheeler condition (3.29) and the Nordtvedt-Pagels condition (3.30), which determine isotropic electromagnetic fields.

Both metrics (3.35) and (3.36) are interesting due to their physical meaning: in such a space, the space curvature is due to an isotropic electromagnetic field. Moreover, if we remove this field from the space, the space becomes flat.

There are also numerous other metrics which are exact solutions to the Einstein-Maxwell equations, related to the class of isotropic electromagnetic fields. However no one of them satisfies Zelmanov's criterion and Lichnérowicz' criterion.

Minkowski's signature permits only two metrics for non-simple recursion curvature spaces. These are the metric

$$\left.\begin{aligned} ds^2 &= \psi\,(x^0, x^2, x^3)(dx^0)^2 + 2\,dx^0 dx^1 + \\ &\quad + K_{22}(dx^2)^2 + 2K_{23}\,dx^2 dx^3 + K_{33}(dx^3)^2 \\ K_{22} &< 0\,, \qquad K_{22}K_{33} - K_{23}^2 < 0 \end{aligned}\right\}, \tag{3.37}$$

where $\psi = \chi_1(x_0)\big(a_{22}(x^2)^2 + 2a_{23}x^2x^3 + a_{33}(x^3)^2\big) + \chi_2(x^0)x^2 + \chi_3(x^0)x^3$, and the metric

$$\left.\begin{aligned} ds^2 &= 2\,dx^0 dx^1 + \psi\,(x^1, x^2, x^3)(dx^1)^2 + \\ &\quad + K_{22}(dx^2)^2 + 2K_{23}\,dx^2 dx^3 + K_{33}(dx^3)^2 \\ K_{22} &< 0\,, \qquad K_{22}K_{33} - K_{23}^2 < 0 \end{aligned}\right\}, \tag{3.38}$$

where $\psi = \chi_1(x_1)\big(a_{22}(x^2)^2 + 2a_{23}x^2x^3 + a_{33}(x^3)^2\big) + \chi_2(x^1)x^2 + \chi_3(x^1)x^3$. Here a_{ij} and K_{ij} $(i, j = 2, 3)$ are constants.

The metrics (3.37) and (3.38) satisfy the Einstein space condition $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ (3.2) only if $\kappa = 0$ that leads to the relationship

$$K_{33}\,a_{22} + K_{22}\,a_{33} - 2K_{23}\,a_{23} = 0\,. \tag{3.39}$$

The metrics (3.37) and (3.38) are related to the sub-kind N of kind II according to Petrov's classification. It is interesting that the metric (3.38) is stationary and, at the same time, describes "pure gravitational radiation" (in the Lichnérowicz sense).

In a general case, where $R_{\alpha\beta} \neq \kappa g_{\alpha\beta}$, the metrics (3.37) and (3.38) satisfy the Riner-Wheeler condition (3.29) and the Nordtvedt-Pagels condition (3.30). Therefore these metrics are exact solutions to the Einstein-Maxwell equations, which describe both gravitational waves and electromagnetic waves without sources. In this general case both metrics satisfy Zelmanov's criterion and Lichnérowicz' criterion.

§4. The chronometrically invariant criterion for gravitational waves and its link to Petrov's classification. All that has been detailed above represents a generally covariant approach to the gravitational wave problem: the presence of such waves in space does not depend on the frame of reference of the observer. There is also another approach to the gravitational wave problem. It determines not only gravitational waves (they are derived from masses), but also gravitational inertial waves (derived from the fields of rotation), both in a frame of reference connected to a real observer. This approach is due to Zelmanov's mathematical apparatus of chronometric invariants [16, 17], which are physically observable quantities in the basic space (space-time) of General Relativity.

In all experimental tests of the General Theory of Relativity, the most important fact is that any real observer, who processes the measurements, rests with respect to his laboratory reference frame and all physical standards located in it. In other words, he is located in a reference frame which accompanies his physical standards (the body of reference). Zelmanov [16, 17] showed that quantities measured by the observer in the accompanying reference frame possess the property of *chronometric invariance*: they are invariant along the three-dimensional section determined by the observer's reference frame (along his three-dimensional space). Keeping this fact in mind, Zelmanov formulated a *chronometrically invariant criterion* for gravitational waves. This criterion is invariant only for the transformations of that reference frame, which rests with respect to the observer and his laboratory references. Following this way, in contrast to the generally covariant approach, we can match our theoretical conclusions and the results obtained from real physical experiments.

Zelmanov showed that the property of chronometric invariance means invariance with respect to the transformations

$$\left.\begin{aligned} &\widetilde{x}^0 = \widetilde{x}^0(x^0, x^1, x^2, x^3) \\ &\widetilde{x}^i = \widetilde{x}^i(x^1, x^2, x^3)\,, \qquad \frac{\partial \widetilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\}, \tag{4.1}$$

then he proved that chronometrically invariant quantities are the respective projections of four-dimensional (generally covariant) quantities onto the line of time and the spatial section of the observer. He had developed a versatile mathematical apparatus, which allows one to derive the chronometrically invariant projections from any generally covariant quantities (and equations) and is known as the *theory of chronometric invariants*. The core of the theory and necessary details were presented by him in the publications [16, 17].

In the framework of the theory, a chronometrically invariant d'Alembert operator was introduced as

$$^*\Box = h^{ik}\,{}^*\nabla_i\,{}^*\nabla_k - \frac{1}{a^2}\frac{^*\partial^2}{\partial t^2}\,, \tag{4.2}$$

where $h^{ik} = -\,g^{ik}$ is the chr.inv.-metric tensor presented in its contravariant form (its contravariant, covariant, and mixed forms differ, see below), $^*\nabla_i$ is the symbol of chr.inv.-differentiation (a chr.inv.-analogue to the symbol ∇_σ of generally covariant differentiation), a is the linear velocity at which the attraction of gravity spreads, $\frac{^*\partial}{\partial t}$ is the chr.inv.-differential operator with respect to time.

This is Zelmanov's chronometrically invariant criterion for gravitational waves and gravitational inertial waves:

Zelmanov's chr.inv.-criterion. If the metric of a space possesses wave properties, the chr.inv.-quantities f, characterizing the local reference space of an observer, such as the gravitational inertial force F_i, the angular velocity of the rotation of the space A_{ik}, the deformation tensor D_{ik}, the spatial curvature tensor C_{iklj} (also the scalar quantities, derived from them), and the chr.inv.-projections X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor must satisfy the chr.inv.-d'Alembert equation

$$^*\Box\, f = A\,, \tag{4.3}$$

where A is an arbitrary function of the four-dimensional coordinates, and contains only first derivatives of the chr.inv.-quantities represented by f.

Zelmanov's chr.inv.-criterion is true for the generalized gravitational wave metric (3.27) in the case where the gravitational inertial force F^i is a wave function. At the same time, the generally covariant criteria for gravitational waves are derived from a limitation imposed on the Riemann-Christoffel curvature tensor in order that it be a wave function. Therefore, it would be interesting to study the chr.inv.-components of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ [16]

$$X^{ik} = -\,c^2\,\frac{R_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \qquad Y^{ijk} = -\,c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \qquad Z^{ijkl} = c^2 R^{ijkl} \tag{4.4}$$

in the case, where they are wave functions as well.

What is common among Zelmanov's generally covariant criterion (3.7) and his chr.inv.-criterion (4.3)? The answer to the question arrives from Zelmanov's generally covariant criterion, $\Box^{\sigma}_{\sigma}R_{\mu\alpha\beta\nu}=0$ (3.7), rewritten in chr.inv.-form

$$^*\Box\, X^{ij} = A^{ij}_{(1)}\,, \qquad ^*\Box\, Y^{ijk} = A^{ijk}_{(2)}\,, \qquad ^*\Box\, Z^{iklj} = A^{iklj}_{(3)}\,, \tag{4.5}$$

where $A^{ij}_{(1)}$, $A^{ijk}_{(2)}$, $A^{iklj}_{(3)}$ are chr.inv.-tensors, which contain only first derivatives of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}. From these formulae we arrive at an obvious conclusion, which is:

> Spaces, which satisfy Zelmanov's generally covariant criterion, also satisfy Zelmanov's chr.inv.-criterion. Therefore, the chr.inv.-components of the Riemann-Christoffel curvature tensor play a rôle of wave functions in gravitational wave fields.

Looking at the formula (4.2) of the chr.inv.-d'Alembert operator, together with Zelmanov's chr.inv.-criterion, we see two necessary conditions for *physically observable* gravitational waves:

1) The chr.inv.-quantities f are non-stationary, i.e. $\frac{^*\partial f}{\partial t}\neq 0$;
2) The field of each quantity f is inhomogeneous, i.e. $^*\nabla_i\, f_k \neq 0$.

The wave functions X_{ij}, Y_{ijk}, Z_{iklj} satisfy the requirements only if the observable mechanical characteristics of the observer's reference space (the chr.inv.-quantities F_i, A_{ik}, D_{ik}) and its observable geometric characteristic (the chr.inv.-curvature C_{iklj}) also satisfy them.

When Zelmanov began to construct his cosmological theory of an inhomogeneous anisotropic universe [16], he introduced conditions of the inhomogeneity of a finite region of space. The conditions of inhomogeneity are formulated, in the framework of the chronometrically invariant formalism, as follows [16, 17]

$$^*\nabla_i\, F_k \neq 0\,, \quad ^*\nabla_j\, A_{ik} \neq 0\,, \quad ^*\nabla_j\, D_{ik} \neq 0\,, \quad ^*\nabla_j\, C_{ik} \neq 0\,. \tag{4.6}$$

It is obvious that the wave functions X^{ij}, Y^{ijk}, Z^{iklj}, being taken under these conditions, shall be inhomogeneous as well.

Considering the chr.inv.-formulae of the gravitational inertial force F_i and the angular velocity of the rotation of space A_{ik} [16, 17]

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \mathrm{w} = c^2\,(1-\sqrt{g_{00}}\,)\,, \tag{4.7}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\,(F_i v_k - F_k v_i)\,, \tag{4.8}$$

we see that non-stationary states of a gravitational inertial force field are due to the non-stationarity of its gravitational potential w or the linear velocity v_i of the rotation of space, determined as

$$v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \quad v^i = -c\,g^{0i}\sqrt{g_{00}}\,, \quad v_i = h_{ik}v^k, \quad v^2 = h_{ik}v^iv^k. \tag{4.9}$$

Two fundamental chronometrically invariant identities

$$\left.\begin{aligned} &\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0 \\ &\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_iA_{km} + F_kA_{mi} + F_mA_{ik}\right) = 0 \end{aligned}\right\} \tag{4.10}$$

introduced by Zelmanov (I refer to them as *Zelmanov's identities*), linking F_i and A_{ik}, lead us to the conclusion that the source of non-stationary states of v_i is the vortical nature of the gravitational inertial force F_i (the vorticity means ${}^*\nabla_k F_i - {}^*\nabla_i F_k \neq 0$).

The cause of non-stationary states of the deformation D_{ik} of space, which is determined in chr.inv.-form as [16, 17]

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D = h^{ik}D_{ik} = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \tag{4.11}$$

where $h = \det\|h_{ik}\|$, is the non-stationarity of the physically observable metric tensor h_{ik}, determined by Zelmanov [16, 17] as

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}} = -\,g_{ik} + \frac{1}{c^2}\,v_iv_k\,, \quad h^{ik} = -\,g^{ik}, \quad h^i_k = \delta^i_k\,. \tag{4.12}$$

The non-stationarity of the chr.inv.-metric tensor h_{ik} is also the cause of non-stationary states of the chr.inv.-curvature

$$C_{lkij} = H_{lkij} - \frac{1}{c^2}\left(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + {} + A_{kl}D_{ij} + A_{li}D_{jk}\right) \tag{4.13}$$

and the chr.inv.-quantities $C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im}C_{kimj}$ and $C = C^j_j = h^{lj}C_{lj}$ derived from it (the chr.inv.-scalar C is the *three-dimensional observable curvature*). They are determined through the Schouten chr.inv.-tensor $H_{lki\cdot}^{\cdots j}$ and the Christoffel chr.inv.-symbols Δ^k_{ij}

$$H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial \Delta^j_{kl}}{\partial x^i} - \frac{{}^*\partial \Delta^j_{il}}{\partial x^k} + \Delta^m_{kl}\Delta^j_{im} - \Delta^m_{il}\Delta^j_{km}\,, \tag{4.14}$$

$$\Delta^k_{ij} = h^{km}\Delta_{ij,m} = \frac{1}{2}\,h^{km}\left(\frac{{}^*\partial h_{im}}{\partial x^j} + \frac{{}^*\partial h_{jm}}{\partial x^i} - \frac{{}^*\partial h_{ij}}{\partial x^m}\right), \tag{4.15}$$

which are Zelmanov's remakes [16,17] of Schouten's tensor and Christoffel's symbols according to the chronometrically invariant formalism.

Here $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}v_i\frac{^*\partial}{\partial t}$ are the chr.inv.-differential operators with respect to time and the spatial coordinates.

Zelmanov [16] had obtained how the chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ are expressed through the (observable) chr.inv.-characteristics of space. These formulae, having indices lowered by the chr.inv.-metric tensor h_{ik}, are

$$X_{ij} = \frac{^*\partial D_{ij}}{\partial t} - \left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D_{jl} + A_{jl}\right) + \frac{1}{2}\left(^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}F_i F_j\,, \tag{4.16}$$

$$Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2}A_{ij}F_k\,, \tag{4.17}$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2 C_{iklj}\,. \tag{4.18}$$

We see from here that non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} are due to the non-stationarity of the chr.inv.-characteristics of space (F_i, A_{ik}, D_{ik}, C_{iklj}), thus — the non-stationarity of the components of the fundamental metric tensor $g_{\alpha\beta}$, namely

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2, \quad g_{0i} = -\frac{1}{c}\,v_i\left(1 - \frac{\mathrm{w}}{c^2}\right), \quad g_{ik} = -h_{ik} + \frac{1}{c^2}\,v_i v_k\,. \tag{4.19}$$

We consider each of these cases here, mindful of the need to find theoretical grounds for the gravitational wave problem:

1) Non-stationary states of the time component g_{00} derive from the time variation of the gravitational potential w;
2) Non-stationary states of the mixed components g_{0i} derive from the non-stationarity of the rotation of space or the gravitational potential w (or from both these factors);
3) Non-stationary states of the spatial components g_{ik} derive from the aforementioned two factors as well.

The metric of weak plane gravitational waves has the form

$$ds^2 = c^2dt^2 - (dx^1)^2 - (1+a)(dx^2)^2 + 2b\,dx^2dx^3 - (1-a)(dx^3)^2, \tag{4.20}$$

where $a = a(ct + x^1)$ and $b = b(ct + x^1)$ if the wave travels in the direction x^1, and they are small values.

As seen, in this metric there is not a gravitational potential ($\mathrm{w} = 0$) as soon as there is not rotation of space ($v_i = 0$). For this reason we arrive at a very important conclusion:

> Weak plane gravitational waves are derived from sources other than gravitational fields of masses.

An analogous situation arises in relativistic cosmology, where, until this day, the main rôle is played by the theory of a homogeneous isotropic universe. This theory is based on the metric of a homogeneous isotropic space (see Chapter 1 in [16], for detail)

$$\left.\begin{aligned} & ds^2 = c^2dt^2 - R^2\frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\left[1 + \frac{k}{4}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right]\right]^2} \\ & R = R\,(t)\,, \quad k = 0, \pm 1 \end{aligned}\right\}. \tag{4.21}$$

When one substitutes this metric into Einstein's equations, one obtains a spectrum of solutions, which are known as homogeneous isotropic models, or the *Friedmann cosmological models* [16].

Taking our previous conclusion on the origin of weak plane gravitational waves into account, we come to another important conclusion:

> No gravitational wave fields derived from masses can exist in any Friedmann universe. Moreover, any Friedmann universe is free of gravitational inertial waves derived from the fields of rotation.

Currently there is not indubitable observational data supporting the absolute rotation of the Universe. This problem is under considerable discussion among astronomers and physicists over decades, and remains open. Rotations of bulky space bodies like planets, stars, and galaxies are beyond any doubt. But these rotations do not result from the absolute rotation of the whole Universe, including the absolute rotation of its common gravitational field.

Looking back at the question of whether or not gravitational waves and gravitational inertial waves exist, or whether or not non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} exist, we conclude that non-stationary states of the wave functions are derived from:

1) The case, where the field of the acting gravitational inertial force F_i is vortical (the non-stationarity of g_{00} and g_{0i});
2) Non-stationary states of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$.

In the first case, the effect of gravitational inertial waves or gravitational inertial waves manifests itself as non-stationary corrections to the clock of the observer. In the second case, the proper time of the observer flows unchanged, while gravitational waves or gravitational inertial waves are presented as waves of only the deformation of space.

My task herein is to construct basics of the chronometrically invariant theory of gravitational waves and gravitational inertial waves.

It is possible to show that the chr.inv.-components of the Riemann-Christoffel curvature tensor, which are the wave functions X^{ij}, Y^{ijk}, Z^{iklj}, possess the properties

$$X_{ij} = X_{ji}\,, \quad X^k_k = -\kappa c^2, \quad Y_{[ijk]} = 0\,, \quad Y_{ijk} = -Y_{ikj}\,. \tag{4.22}$$

Equations (4.4) being taken in an ortho-frame (where $g_{00}=1$, $g_{0i}=0$, and $g_{ik}=\delta_{ik}$, thus there is not difference between the covariant and contravariant components of a tensor) take the form

$$X_{ij} = -c^2 R_{0i0j}\,, \quad Y_{ijk} = -c R_{0ijk}\,, \quad Z_{iklj} = c^2 R_{iklj}\,. \tag{4.23}$$

Once we re-write the Einstein space condition $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ (3.2) in an ortho-frame, we take the formulae (4.23) into account. Then, introducing three-dimensional matrices x and y such that

$$x \equiv \|x_{ik}\| = -\frac{1}{c^2}\,\|X_{ik}\|\,, \qquad y \equiv \|y_{ik}\| = -\frac{1}{2c}\,\|\varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn}\|\,, \tag{4.24}$$

where ε_{imn} is the three-dimensional completely antisymmetric discriminant chr.inv.-tensor, we compose a six-dimensional matrix $\|R_{ab}\|$

$$\|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \qquad a, b = 1, 2, \dots 6\,, \tag{4.25}$$

which satisfies the conditions

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad y_{11} + y_{22} + y_{33} = 0\,. \tag{4.26}$$

Now, let us compose a lambda-matrix

$$\|R_{ab} - \Lambda g_{ab}\| = \begin{Vmatrix} x + \Lambda\varepsilon & y \\ y & -x - \Lambda\varepsilon \end{Vmatrix}, \tag{4.27}$$

where ε is the three-dimensional unit matrix. After elementary transformations, we reduce this lambda-matrix to the form

$$\begin{Vmatrix} x + iy + \Lambda\varepsilon & 0 \\ 0 & -x - iy - \Lambda\varepsilon \end{Vmatrix} = \begin{Vmatrix} \bar{Q}\,(\Lambda) & 0 \\ 0 & \bar{Q}\,(\Lambda) \end{Vmatrix}. \tag{4.28}$$

As is known according to Petrov [27], the initial lambda-matrix can have only one of characteristics drawn from three kinds: I) $[111, \overline{111}]$; II) $[21, \overline{21}]$; III) $[3, 3]$. Using, according to Petrov, the canonical form of the matrix $\|R_{ab}\|$ in a non-holonomic ortho-frame for each of these three kinds of the curvature tensor, we express the matrix $\|R_{ab}\|$ through the chr.inv.-tensors X_{ij} and Y_{ijk}. We obtain, for kind I,

$$\left.\begin{array}{c}\underline{\text{Kind I}}\\[4pt] \|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \\[8pt] x = \begin{Vmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{Vmatrix}, \quad y = \begin{Vmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{Vmatrix}\end{array}\right\}, \tag{4.29}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad y_{11} + y_{22} + y_{33} = 0\,. \tag{4.30}$$

Using (4.24) we also express the stationary curvatures Λ_i (3.21) $(i = 1, 2, 3)$ through X_{ij} and Y_{ijk}

$$\left.\begin{aligned} \Lambda_1 &= -\frac{1}{c^2}\,X_{11} + \frac{i}{c}\,Y_{123} \\ \Lambda_2 &= -\frac{1}{c^2}\,X_{22} + \frac{i}{c}\,Y_{231} \\ \Lambda_3 &= -\frac{1}{c^2}\,X_{33} + \frac{i}{c}\,Y_{312} \end{aligned}\right\}. \tag{4.31}$$

Hence the chr.inv.-quantities X_{ik} consist of the real parts of the stationary curvatures Λ_i (the term α_i in 3.21), while the chr.inv.-quantities Y_{ijk} consist the imaginary parts (the term $i\beta_i$ in formula 3.21). In spaces of the sub-kind D $(\Lambda_2 = \Lambda_3)$ we have: $X_{22} = X_{33}$, $Y_{231} = Y_{312}$. In spaces of the sub-kind O $(\Lambda_1 = \Lambda_2 = \Lambda_3)$ we have: $X_{11} = X_{22} = X_{33} = -\frac{\kappa c^2}{3}$, $Y_{123} = Y_{231} = Y_{312} = 0$. Hence Einstein spaces of the sub-kind O have only real curvatures, while being empty they are flat.

For kind II we have

$$\left.\begin{array}{c}\underline{\text{Kind II}}\\[4pt] \|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \\[8pt] x = \begin{Vmatrix} x_{11} & 0 & 0 \\ 0 & x_{22}+1 & 0 \\ 0 & 0 & x_{22}-1 \end{Vmatrix}, \quad y = \begin{Vmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 1 \\ 0 & 1 & y_{22} \end{Vmatrix}\end{array}\right\}, \tag{4.32}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad x_{22} - x_{33} = 2\,, \quad y_{11} + 2y_{22} = 0\,. \tag{4.33}$$

The stationary curvatures in this case are

$$\left.\begin{aligned} \Lambda_1 &= -\frac{1}{c^2}\,X_{11} + \frac{i}{c}\,Y_{123} \\ \Lambda_2 &= -\frac{1}{c^2}\,X_{22} - 1 + \frac{i}{c}\,Y_{231} \\ \Lambda_3 &= -\frac{1}{c^2}\,X_{33} + 1 + \frac{i}{c}\,Y_{312} \end{aligned}\right\}. \tag{4.34}$$

From these results we conclude that the stationary curvatures Λ_2 and Λ_3 can never become zero in this case, so Einstein spaces (gravitational fields) of kind II are curved in any case. They cannot approach a flat space.

In spaces of kind II ($\Lambda_1 = \Lambda_2 = 0$; if this is the sub-kind N of kind II, there is also $\kappa = 0$), in an ortho-frame, we have

$$X_{11} = X_{22} - \kappa c^2 = X_{33} + \kappa c^2, \qquad Y_{123} = Y_{231} = Y_{312} = 0\,, \tag{4.35}$$

so the stationary curvatures take real numerical values. In an empty space of this kind, the matrices x and y are degenerate (determinants of these matrices are zero). For this reason spaces of the sub-kind N of kind II are *degenerate*. Thus, I refer to gravitational fields which fill spaces of the sub-kind N of kind II as *degenerate gravitational fields*. In emptiness ($\kappa = 0$) several elements of the matrices x and y take the numerical values $+1$ and -1 thereby making an ultimate transition to a flat space impossible.

For kind III we have

$$\left.\begin{gathered} \underline{\text{Kind III}} \\ \|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \\ x = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad y = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix} \end{gathered}\right\}. \tag{4.36}$$

Here the stationary curvatures are zero and both the matrices x and y are degenerate. Einstein spaces of kind III can only be empty ($\kappa = 0$), but, at the same time, they can never be flat.

These are the basics of the chronometrically invariant theory of gravitational waves and gravitational inertial waves, which I have introduced in this paragraph for the case of empty Einstein spaces. Numerous important conclusions follow from the theory.

The conclusions are related to the (observable) chr.inv.-components X^{ik}, Y^{ijk}, Z^{ijkm} of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$, which are wave functions in a wave gravitational field. Further, I will refer to the chr.inv.-components according to their physical meaning:

1) X^{ik}, as a projection onto the line of time, manifests the variation of the curvature tensor with time at the same location. This is the *stationary observable component* of the curvature tensor;
2) Y^{ijk} is a mixed (space-time) projection. It manifests a shift of the time variation of the curvature tensor with the variation of the three-dimensional (spatial) coordinates. This is the *dynamical observable component* of the curvature tensor. This is a "truly gravitational wave component", which, being nonzero ($Y^{ijk} \neq 0$), manifests the presence of gravitational waves or gravitational inertial waves travelling in space;
3) Z^{ijkm}, which is a purely spatial projection, is an "instant three-dimensional shot" (or "section") of the curvature tensor. This is the *distributive observable component.*

Proceeding from the equations deduced for the canonical form of the matrix $\|R_{ab}\|$, obtained in the framework of the chr.inv.-theory, we conclude:

> The dynamical observable component Y^{ijk} of the curvature tensor can be zero ($Y^{ijk} = 0$) only in spaces of kind I (the stationary curvatures take real values in this case). Moreover, $Y^{ijk} = 0$ in all known metrics of kind I. Gravitational fields of spaces of kind I are derived from islands of mass located in emptiness. Thus, gravitational waves and gravitational inertial waves cannot derive from islands of mass located in an empty space (at least, in the framework of all known metrics of kind I).

In particular, this means that search for gravitational radiation, targeting rotating cosmic bodies in emptiness as its source, cannot be a proper experimental test to the General Theory of Relativity.

According to most of the gravitational wave criteria, the presence of gravitational waves is linked to spaces of the sub-kind N of kind II, and kind III, where the matrix y_{ik} has components equal to $+1$ or -1. Moreover, in the fields of the sub-kind N of kind II, and kind III, the numerical values $+1$ or -1 are attributed also to components of the matrix x_{ik}. This implies that:

> Spaces, which contain gravitational fields satisfying the gravitational wave criteria (these are spaces of the sub-kind N of kind II,

and kind III), are curved independently of whether or not they are empty ($R_{\alpha\beta}=0$) or filled with distributed matter ($R_{\alpha\beta}=\kappa g_{\alpha\beta}$). In any case, in these spaces gravitational radiation is derived from the "interaction" between the stationary observable component X^{ij} and the dynamical observable component Y^{ijk} of the curvature tensor, which are nonzero therein.

Petrov's classification of spaces (gravitational fields) applied here to the gravitational wave problem is valid only to Einstein spaces. Solving this problem for spaces of a general kind, where $R_{\alpha\beta}\neq\kappa g_{\alpha\beta}$, is a highly complicate task due to some mathematical difficulties. Namely, when having an arbitrary distribution of matter in a space, the matrix $\|R_{ab}\|$, taken in a non-holonomic ortho-frame, is not symmetrically doubled; on the contrary, the matrix takes the form

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y' & z \end{matrix} \right\|, \tag{4.37}$$

where the three-dimensional matrices x, y, z are constructed on the following elements, respectively*

$$\left.\begin{aligned} x_{ik} &= -\frac{1}{c^2}\,X_{ik} \\ z_{ik} &= \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq} \\ y_{ik} &= \frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn} \end{aligned}\right\}, \tag{4.38}$$

and y' means transposition. It is obvious that reducing this matrix to its canonical form will meet severe mathematical difficulties, thus becoming a highly complicate task.

Nevertheless Petrov's classification, which has successfully been applied here to the chr.inv.-theory of gravitational waves and gravitational inertial waves, allows us to conclude:

The stationary observable component X^{ij} and the dynamical observable component Y^{ijk} of the curvature tensor are different in their physical origin†. Space metrics can exist even in a case, where

*In ortho-frames there is not difference between the covariant and contravariant components of a tensor [27]. Therefore, we can replace $z_{ik}=\frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq}$ and $y_{ik}=\frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}$ with $z_{ik}=\frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z_{mnpq}$ and $y_{ik}=\frac{1}{2c}\,\varepsilon_{imn}Y_{kmn}$ in (4.38). This can also be applied to the equations of formula (4.24).

†We do not discuss the spatial observable component Z^{iklj}, because, in an ortho-frame, the matrices x and z are connected by the ratio $x=-z$. Therefore, the components X^{ik} and Z^{iklj} are connected to each other in this case.

$Y^{ijk}=0$ but $X^{ij}\neq 0$ and $Z^{iklj}\neq 0$ (these are spaces of kind I). However, among all known solutions of Einstein's equations, there is not a metric for which $Y^{ijk}\neq 0$ but $X^{ij}=0$ and $Z^{iklj}=0$. Therefore, in gravitational wave fields and gravitational inertial wave fields, $Y^{ijk}\neq 0$ and $X^{ij}\neq 0$ (and $Z^{iklj}\neq 0$ as well: see the footnote on page 56) everywhere and always.

§5. Physical conditions of the existence of gravitational waves in non-empty spaces. In §4, I suggested a chr.inv.-theory of gravitational waves and gravitational inertial waves for empty Einstein spaces. Now, I extend the theory to non-empty Einstein spaces.

As was shown in §4, in the framework of the chr.inv.-theory of gravitational waves and gravitational inertial waves, the necessary condition of the existence of the waves are the inhomogeneity and non-stationarity of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}, which are the observable components of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$. The conditions of homogeneity in the presence of distributed matter (medium) are formulated, in the framework of the chronometrically invariant formalism [16], as follows

$$\left.\begin{array}{l} {}^*\nabla_i F_k = 0\,, \quad {}^*\nabla_j A_{ik} = 0\,, \quad {}^*\nabla_j D_{ik} = 0\,, \quad {}^*\nabla_j C_{ik} = 0 \\ \dfrac{{}^*\partial\rho}{\partial x^i} = 0\,, \quad {}^*\nabla_j J_i = 0\,, \quad {}^*\nabla_j U_{ik} = 0 \end{array}\right\}, \qquad (5.1)$$

where ρ, $J_i = h_{ik}J^k$, and $U_{ik} = h_{im}h_{kn}U^{mn}$ are the observable density of matter, the observable density of momentum, and the observable stress tensor, which are the respective chr.inv.-projections

$$\rho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{cT_0^i}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik} \qquad (5.2)$$

of the energy-momentum tensor $T_{\alpha\beta}$ of the matter (from which we can also obtain $U = h^{ik}U_{ik}$).

Once the conditions of inhomogeneity (5.1) are satisfied, the wave functions represented by f in Zelmanov's chr.inv.-criterion for gravitational waves and gravitational inertial waves (4.3) are homogeneous as well, thus the d'Alembertian (4.3) becomes trivial.

Now, let us study the conditions of the non-stationarity of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} in the presence of a distributed matter. To do it, we should express them through the chr.inv.-characteristics of the matter. We will use Einstein's equations and also the conservation law of the energy-momentum tensor, written in chr.inv.-form. In [16], Zelmanov considered Einstein's generally covariant equations (3.1) in the general case, where any kind of distributed matter is presented: the formula for the energy-momentum tensor $T_{\alpha\beta}$ is not detailed there. According to Zelmanov, they have chr.inv.-projections as follows (I refer to them as the *Einstein chr.inv.-equations*)

$$\frac{{}^*\partial D}{\partial t}+D_{jl}D^{jl}+A_{jl}A^{lj}+{}^*\nabla_j F^j-\frac{1}{c^2}F_jF^j=-\frac{\varkappa}{2}\left(\rho c^2+U\right)+\lambda c^2, \quad (5.3)$$

$${}^*\nabla_j\left(h^{ij}D-D^{ij}-A^{ij}\right)+\frac{2}{c^2}F_jA^{ij}=\varkappa J^i, \quad (5.4)$$

$$\frac{{}^*\partial D_{ik}}{\partial t}-\left(D_{ij}+A_{ij}\right)\left(D^j_k+A^{\cdot j}_{k\cdot}\right)+DD_{ik}+3A_{ij}A^{\cdot j}_{k\cdot}+$$
$$+\frac{1}{2}\left({}^*\nabla_iF_k+{}^*\nabla_kF_i\right)-\frac{1}{c^2}F_iF_k-c^2C_{ik}=$$
$$=\frac{\varkappa}{2}\left(\rho c^2h_{ik}+2U_{ik}-Uh_{ik}\right)+\lambda c^2h_{ik}\,, \quad (5.5)$$

where ${}^*\nabla_i$ is the symbol of chr.inv.-differentiation (a chr.inv.-analogue to the symbol ∇_σ of generally covariant differentiation). He also considered the general covariant conservation law equation

$$\nabla_\sigma T^{\alpha\sigma}=0 \quad (5.6)$$

of the energy-momentum tensor (also in the general case of arbitrary matter). It has the following chr.inv.-projections [16]

$$\frac{{}^*\partial\rho}{\partial t}+D\rho+\frac{1}{c^2}D_{ij}U^{ij}+{}^*\nabla_iJ^i-\frac{2}{c^2}F_iJ^i=0\,, \quad (5.7)$$

$$\frac{{}^*\partial J^k}{\partial t}+DJ^k+2\left(D^k_i+A^{\cdot k}_{i\cdot}\right)J^i+{}^*\nabla_iU^{ik}-\frac{2}{c^2}F_iU^{ik}-\rho F^k=0\,. \quad (5.8)$$

We begin the study from the simplest case, where all kinematic characteristics of a non-empty space are zero. In this case, the reference frame of the observer (his local space of reference) falls freely, is free of rotation, and does not deform. In other words,

$$F_i=0\,,\qquad A_{ik}=0\,,\qquad D_{ik}=0\,, \quad (5.9)$$

thus the chr.inv.-components of the curvature tensor (the wave functions) take the form

$$X^{ik}=0\,,\qquad Y^{ijk}=0\,,\qquad Z^{iklj}=-c^2C^{iklj}. \quad (5.10)$$

It is easy to see that, in this case, the solely nonzero component Z^{iklj}

of the curvature tensor is stationary. Therefore, gravitational waves and gravitational inertial waves are impossible in this case.

Construct the metric of a respective space (space-time) for this case. The conditions $F_i = 0$ and $A_{ik} = 0$ mean, respectively, that $g_{00} = 1$ and $g_{0i} = 0$ in the space. The fact that the space does not deform ($D_{ik} = 0$) points to the stationarity of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$. According to Cotton [45], in this case the three-dimensional metric can be reduced to diagonal form. Therefore, a space which satisfies the physical conditions (5.9) is a reducible space, whose metric takes the form

$$ds^2 = c^2 dt^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \tag{5.11}$$

where the components g_{ii} do not depend on time.

Thus, we arrive at the following obvious conclusion:

> In non-empty spaces, whose all kinematic characteristics are zero, gravitational waves and gravitational inertial waves are impossible due to the stationarity of all the chr.inv.-components of the curvature tensor (the wave functions of space).

Consider another kind of non-empty spaces, which do not contain fields of acceleration (the gravitational potential is homogeneously distributed therein), do not deform, but rotate. A typical instance of such spaces are those described by Gödel's metric [46], where

$$F_i = 0\,, \qquad D_{ik} = 0\,, \qquad A_{ik} \neq 0\,. \tag{5.12}$$

The first condition of these, $F_i = 0$, according to the chronometrically invariant formalism, means

$$g_{00} = 1\,, \qquad \frac{{}^*\partial g_{0i}}{\partial t} = 0\,, \tag{5.13}$$

therefore the rotation of a Gödel space is stationary. Because the chr.inv.-metric tensor has the form $h_{ik} = -\,g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}$, we see that g_{ik} does not depend on time in this case. This means, being applied to the wave functions X^{ik}, Y^{ijk}, Z^{iklj}, that

$$\frac{{}^*\partial X^{ik}}{\partial t} = 0\,, \qquad \frac{{}^*\partial Y^{ijk}}{\partial t} = 0\,, \qquad \frac{{}^*\partial Z^{iklj}}{\partial t} = 0\,, \tag{5.14}$$

i.e. Gödel's metric is completely stationary. Hence,

> In non-empty spaces, which do not deform but rotate with a constant linear velocity, gravitational waves and gravitational inertial waves are impossible.

Therefore, in searching for gravitational radiation, where binary stars are targeted as its source, we should focus onto only those binaries, whose rotation is non-stationary. In particular, if a satellite-star decelerates (due to some reasons) when orbiting the main star, the binary system should emit gravitational radiation.

Now, consider that case of non-empty spaces, where spaces do not deform, do not rotate, but contain fields of acceleration (the gravitational inertial force is nonzero therein). In this case,

$$F_i \neq 0\,, \qquad A_{ik} = 0\,, \qquad D_{ik} = 0\,. \tag{5.15}$$

The condition $A_{ik}=0$ means $g_{0i}=0$. The condition $D_{ik}=0$, as was explained above, means that the observable metric h_{ik} of the space is stationary, hence g_{ik} does not depend on time: in this case, according to Cotton [45], the three-dimensional metric can be transformed to diagonal form. Finally, the metric of such a space takes the form

$$ds^2 = g_{00}\,(ct, x^1, x^2, x^3)\,c^2dt^2 + g_{ii}\,(x^1, x^2, x^3)(dx^i)^2, \tag{5.16}$$

so the chr.inv.-components of the curvature tensor take the form

$$\left.\begin{aligned} X_{ik} &= \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}\,F_iF_k \\ Y^{ijk} &= 0\,, \qquad Z_{iklj} = -\,c^2 C_{iklj} \end{aligned}\right\}. \tag{5.17}$$

Due to the absence of the rotation and deformation, the wave function Z^{iklj} is stationary. So, only the non-stationarity of the wave function X^{ik} can be supposed. Using the Einstein chr.inv.-equations while taking the physical conditions (5.15) into account, we express X^{ik} (5.17) through the chr.inv.-characteristics of the distributed matter

$$X_{ik} = c^2C_{ik} + \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \lambda c^2 h_{ik}\,. \tag{5.18}$$

This, however, does not matter in this case. Anyhow, due to the fact that the dynamical observable component Y^{ijk} of the curvature tensor is zero in such spaces, we immediately arrive at the folowing conclusion:

> In non-empty spaces, which contain fields of the gravitational inertial force, but are free of rotation and deformation, gravitational waves and gravitational inertial waves are impossible.

Now, the last case of non-empty spaces remains under focus. In this case, the space does not deform, but rotates and contains the field of the gravitational inertial force

$$F_i \neq 0\,, \qquad A_{ik} \neq 0\,, \qquad D_{ik} = 0\,. \tag{5.19}$$

Running ahead of the obtained result, I announce that this is the most interesting case of non-deforming non-empty Einstein spaces, because it permits gravitational radiation.

The wave functions in this case take the form

$$X^{ik} = 3A^{i}_{\cdot j}A^{kj} - c^2 C^{ik} + \frac{\varkappa}{2}\left(\rho c^2 h^{ik} + 2U^{ik} - Uh^{ik}\right) + \lambda h^{ik}, \quad (5.20)$$

$$Y^{ijk} = {}^*\nabla^j A^{ik} - {}^*\nabla^i A^{jk} + \frac{2}{c^2}\, A^{ji}F^k, \quad (5.21)$$

$$Z^{iklj} = A^{ik}A^{lj} - A^{il}A^{kj} + 2\,A^{ij}A^{kl} - c^2 C^{iklj}. \quad (5.22)$$

Analyzing the formulae, we apply Zelmanov's 1st identity (4.10), which links the non-stationarity of A_{ik} to the vortex of F_i. We take into account that $\frac{{}^*\partial A_{ik}}{\partial t} = h_{im}h_{kn}\frac{{}^*\partial A^{mn}}{\partial t}$ in non-deforming spaces. We obtain that: 1) the non-stationarity of X^{ik} can be due to the vortex of the gravitational inertial force F_i, the non-stationarity of the factors of the observable three-dimensional curvature C^{ik}, the observable components of the energy-momentum tensor, and the cosmological term, or due to all these factors; 2) the non-stationarity of Y^{ijk} can only be due to the common presence of the vortex of the field F_i and the non-stationarity of the force F_i; 3) the non-stationarity of Z^{iklj} can be due to the vortex of the field F_i or the non-stationarity of the observable three-dimensional curvature C^{iklj}, or due to both these factors.

As was explained in §4, page 55, the dynamical observable component Y^{ijk} of the Riemann-Christoffel curvature tensor is a "truly gravitational wave component", which manifests the presence of gravitational waves or gravitational inertial waves travelling in space. The fact that $Y^{ijk} \neq 0$ in spaces of this kind means that gravitational waves and gravitational inertial waves are possible therein.

Because $Y^{ijk} \neq 0$ (5.21) in the case, we obtain $J^i \neq 0$ from the Einstein chr.inv.-vectorial equation (5.4), and $\frac{{}^*\partial\rho}{\partial t} \neq 0$ due to the chr.inv.-scalar conservation equation (5.7).

The first result, $J^i \neq 0$, implies the presence of a flow of energy-momentum of the medium that fills the space. In other word, the observer (and his frame of reference) does not accompany the medium, but moves with respect to it. As was already shown in §2, the travelling rays of gravitational radiation in emptiness are isotropic geodesics (the rays of the light's travel). Hence, gravitational wave fields and gravitational inertial wave fields are non-isotropic in spaces of this kind: the waves travel at another velocity than light, depending on the properties of the medium.

The second result, $\frac{{}^*\partial\rho}{\partial t} \neq 0$, means that the density of the medium does not remain stationary, but changes with time according to the transit of gravitational waves and gravitational inertial waves. In a barotropic medium, as we know, $p = p(\rho)$ is true. Therefore, if a space of this kind is filled with a barotropic medium, gravitational waves and gravitational inertial waves travelling therein are linked to the non-stationarity of the pressure. If a space of this kind is filled with a barocline medium (it is characterized by the condition $p = p(\rho, T)$, where T is the absolute temperature of the medium), gravitational waves and gravitational inertial waves are linked to the non-stationarity of the pressure and temperature.

Thus, concerning non-empty spaces characterized by the physical conditions (5.19), we conclude:

> Non-empty spaces, which do not deform, but rotate and contain fields of the gravitational inertial force, gravitational waves and gravitational inertial waves are possible. In a barotropic medium, the waves are linked to the non-stationarity of the pressure, while in a barocline medium they are linked to the non-stationary of the pressure and temperature. The waves travel with a velocity different than that of light, depending on the properties of the medium that fills the space.

An important note should be said in the end. When we considered the physical conditions of the existence of gravitational waves in non-empty spaces, we meant that the spaces do not deform ($D_{ik} = 0$). This has been the main assumption and task of this study. As a matter of fact, gravitational waves and gravitational inertial waves can exist in deforming spaces as waves of the space deformation. Therefore, all that has been obtained in this paragraph is related only to non-deforming spaces. The main result obtained herein is:

> It is not necessary that only the deformation of space is the source of gravitational waves and gravitational inertial waves. The waves can exist even in non-deforming spaces, if the gravitational inertial force F_i and the rotation of space A_{ik} differ from zero, and the field F_i is vortical (that means the non-stationarity of A_{ik}).

§6. Chronometrically invariant representation of Petrov's classification for non-empty spaces. In §4, I suggested a chr.inv.-theory of gravitational waves and gravitational inertial waves in empty Einstein spaces. The geometrical structure of Einstein spaces of all three kinds was presented in terms of chronometric invariants. This study was extended to non-empty Einstein spaces in §5: physical conditions of the existence of gravitational radiation in medium were discussed. Now, I shall obtain chr.inv.-components of Weyl's conformal curvature tensor, and find their connexion with the chr.inv.-components of the Riemann-Christoffel curvature tensor. The main task of this deduction is understanding the rôle of matter in forming gravitational radiation in non-empty non-Einstein spaces.

Petrov's classification of spaces where $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ (Einstein spaces) was a resolvable mathematical problem, because the matrix $\|R_{ab}\|$ of the Riemann-Christoffel curvature tensor in an ortho-frame of a six-dimensional Riemannian space is symmetrically doubled due to Einstein's equations. In the case where a space is filled with distributed matter of an arbitrary kind, Einstein's equations manifest that

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y' & z \end{matrix} \right\|, \tag{6.1}$$

where y' is a matrix transposed to the matrix y. This fact makes classification of the curvature tensor in non-empty spaces a very difficult task (see page 56). Therefore, Petrov [27] suggested another solution to this problem. He had constructed a special curvature tensor

$$P_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - S_{\alpha\beta\gamma\delta} + \sigma \left(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma} \right), \tag{6.2}$$

which satisfies all algebraic properties of the Riemann-Christoffel tensor in non-empty spaces, while the additional tensor $S_{\alpha\beta\gamma\delta}$, which takes the energy-momentum tensor (distributed matter) into account, possesses all the properties as well, i.e.

$$S_{\alpha\beta\gamma\delta} = \frac{\varkappa}{2} \left(g_{\alpha\beta}\, T_{\delta\gamma} - g_{\alpha\gamma}\, T_{\beta\delta} + g_{\beta\gamma}\, T_{\alpha\delta} - g_{\beta\delta}\, T_{\alpha\gamma} \right). \tag{6.3}$$

After contraction of the tensor by indices β and δ, and taking Einstein's equations into account, we obtain

$$P_{\alpha\gamma} = (R + 3\sigma)\, g_{\alpha\gamma}\,, \tag{6.4}$$

where σ is a scalar. Once distribution of matter (the energy-momentum tensor $T_{\alpha\beta}$) has been determined, the curvature of space can be found with a precision to within the scalar σ. However the physical meaning of the scalar is still unclear. Therefore, in order to introduce an algebraic classification of non-empty spaces, Weyl's conformal curvature tensor

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2} \left(R_{\alpha\gamma}\, g_{\beta\delta} + R_{\beta\delta}\, g_{\alpha\gamma} - R_{\alpha\delta}\, g_{\beta\gamma} - R_{\beta\gamma}\, g_{\alpha\delta} \right) + \\ + \frac{R}{6} \left(g_{\beta\gamma}\, g_{\alpha\delta} - g_{\beta\delta}\, g_{\alpha\gamma} \right) \tag{6.5}$$

should be applied. This tensor also possesses all the algebraical properties of the Riemann-Christoffel curvature tensor. Also, contracting it by indices β and δ, we obtain

$$C_{\alpha\gamma} = 0\,. \tag{6.6}$$

All these mean that applying Weyl's contracted tensor $C_{\alpha\beta}$ to non-empty spaces, we arrive at an analogy to Ricci's tensor $R_{\alpha\beta}$. Therefore, classification of non-empty non-Einstein spaces according to the algebraic properties of Weyl's conformal curvature tensor $C_{\alpha\beta\gamma\delta}$ should be analogous to Petrov's classification of Einstein spaces. The difference is only that the matrices $\widetilde{x}$ and $\widetilde{y}$ should be used in Weyl's tensor, instead the matrices x and y of the Riemann-Christoffel tensor.

Here we suggest an algebraic classification of Weyl's conformal curvature tensor in terms of chronometric invariants. First, we define the (observable) chr.inv.-components of Weyl's tensor

$$\widetilde{X}^{ik} = -\,c^2\,\frac{C_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \qquad \widetilde{Y}^{ijk} = -\,c\,\frac{C_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \qquad \widetilde{Z}^{iklj} = c^2 C^{ijkl}, \tag{6.7}$$

which are formulated in analogy to those of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ (4.4) as well as those of any 4th rank tensor of the antisymmetric kind as these tensors. The chr.inv.-components (6.7) possess the following properties

$$\widetilde{X}_{ik} = \widetilde{X}_{ki}\,, \quad \widetilde{X}_k^k = 0\,, \quad \widetilde{Y}_{[ijk]} = 0\,, \quad \widetilde{Y}_{ijk} = -\,Y_{ikj}\,, \tag{6.8}$$

where Y_{ikj} is that of $R_{\alpha\beta\gamma\delta}$ (4.4). In an ortho-frame, we have

$$\widetilde{X}_{ik} = -\,c^2\,C_{0i0k}\,, \quad \widetilde{Y}_{ijk} = -\,c\,C_{oijk}\,, \quad \widetilde{Z}_{iklj} = c^2\,C_{iklj}\,. \tag{6.9}$$

Now, we express the chr.inv.-components of Weyl's tensor through the (observable) chr.inv.-characteristics of the distributed matter that fills the space. To do it, we apply the Einstein chr.inv.-equations (they were presented in §5). In an ortho-frame, we obtain

$$C_{0i0k} = -\frac{1}{c^2}\,X_{ik} - \frac{\varkappa}{2c^2}\,U_{ik} + \frac{\varkappa\rho}{6}\,h_{ik} + \frac{\varkappa c^2}{3}\,U h_{ik}\,, \tag{6.10}$$

$$C_{i0jk} = \frac{1}{c}\,Y_{ijk} - \frac{\varkappa}{2\,c}\left(h_{ik}J_i - h_{ij}J_k\right), \tag{6.11}$$

$$C_{iklj} = \frac{1}{c^2}\,Z_{iklj} - \frac{\varkappa}{2c^2}\left(h_{ij}U_{kl} - h_{il}U_{kj} + h_{kl}U_{ij} - h_{kj}U_{il}\right) - \\ - \frac{\varkappa}{3}\left(\rho - \frac{U}{c^2}\right)\left(h_{ik}h_{jl} - h_{il}h_{jk}\right). \tag{6.12}$$

In analogy to (4.24), we introduce three-dimensional matrices

$$\left.\begin{aligned} \widetilde{x} &= \| \widetilde{x}_{ik} \| = -\frac{1}{c^2} \left\| \widetilde{X}_{ik} \right\| \\ \widetilde{y} &= \| \widetilde{y}_{ik} \| = \frac{1}{2c} \left\| \varepsilon_{imn} \widetilde{Y}_{k\cdot\cdot}^{\cdot mn} \right\| \\ \widetilde{z} &= \| \widetilde{z}_{ik} \| = \frac{1}{4c^2} \left\| \varepsilon_{imn} \varepsilon_{kpq} \widetilde{Z}^{mnpq} \right\| \end{aligned}\right\}. \tag{6.13}$$

It is possible to show, from the Einstein equations $C_{\alpha\beta}=0$ written in an ortho-frame, in analogy to Petrov [27] who did it for Einstein's original equations $R_{\alpha\beta}=0$, that $\widetilde{x}_{ik}=-\widetilde{z}_{ik}$. Therefore, we compose a six-dimensional matrix $\|C_{ab}\|$ from Weyl's conformal curvature tensor $C_{\alpha\beta\gamma\delta}$. We obtain a symmetrically paired matrix

$$\|C_{ab}\| = \left\| \begin{matrix} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{matrix} \right\| \tag{6.14}$$

whose elements are connected by the relations

$$\widetilde{x}_{11} + \widetilde{x}_{22} + \widetilde{x}_{33} = 0\,, \qquad \widetilde{y}_{11} + \widetilde{y}_{22} + \widetilde{y}_{33} = 0\,, \tag{6.15}$$

and, as is possible to show, the diagonal components of the matrix $\widetilde{y}$ meet the respective diagonal components of the matrix y.

$$\left.\begin{aligned} \widetilde{y}_{11} &= \frac{1}{c}\,\widetilde{Y}_{123} = \frac{1}{c}\,Y_{123} = y_{11} \\ \widetilde{y}_{22} &= \frac{1}{c}\,\widetilde{Y}_{231} = \frac{1}{c}\,Y_{321} = y_{22} \\ \widetilde{y}_{33} &= \frac{1}{c}\,\widetilde{Y}_{312} = \frac{1}{c}\,Y_{312} = y_{33} \end{aligned}\right\}. \tag{6.16}$$

Composing a lambda-matrix $\|C_{ab}-\Lambda g_{ab}\|$ then reducing it to the canonical form in analogy to Petrov, who did it for the lambda-matrix $\|R_{ab}-\Lambda g_{ab}\|$, we obtain three kinds of non-empty non-Einstein spaces, which are characterized according to Weyl's tensor.

After transformations, we obtain the lambda-matrix $\|C_{ab}-\Lambda g_{ab}\|$ in the form

$$\|C_{ab}-\Lambda g_{ab}\| = \left\| \begin{array}{c|c} \widetilde{x}+i\widetilde{y}+\Lambda\varepsilon & 0 \\ \hline 0 & \widetilde{x}-i\widetilde{y}+\Lambda\varepsilon \end{array} \right\| \equiv$$

$$\equiv \left\| \begin{matrix} Q(\Lambda) & 0 \\ 0 & \bar{Q}\,(\Lambda) \end{matrix} \right\|. \tag{6.17}$$

In analogy to Petrov's classification of the matrix $\|R_{ab}\|$, we obtain, in an ortho-frame, three respective kinds of the matrix $\|C_{ab}\|$

$$\left.\begin{array}{c}\underline{\text{Kind I}}\\ \|C_{ab}\| = \left\| \begin{array}{cc} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{array} \right\|, \\ \widetilde{x} = \left\| \begin{array}{ccc} \widetilde{x}_{11} & 0 & 0 \\ 0 & \widetilde{x}_{22} & 0 \\ 0 & 0 & \widetilde{x}_{33} \end{array} \right\|, \qquad \widetilde{y} = \left\| \begin{array}{ccc} \widetilde{y}_{11} & 0 & 0 \\ 0 & \widetilde{y}_{22} & 0 \\ 0 & 0 & \widetilde{y}_{33} \end{array} \right\| \end{array}\right\}, \tag{6.18}$$

where $\widetilde{x}_{11}+\widetilde{x}_{22}+\widetilde{x}_{33}=0$, $\widetilde{y}_{11}+\widetilde{y}_{22}+\widetilde{y}_{33}=0$ (so in this case there are 4 independent parameters, determining the space structure by an invariant form),

$$\left.\begin{array}{c}\underline{\text{Kind II}}\\ \|C_{ab}\| = \left\| \begin{array}{cc} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{array} \right\|, \\ \widetilde{x} = \left\| \begin{array}{ccc} \widetilde{x}_{11} & 0 & 0 \\ 0 & \widetilde{x}_{22}+1 & 0 \\ 0 & 0 & \widetilde{x}_{22}-1 \end{array} \right\|, \quad \widetilde{y} = \left\| \begin{array}{ccc} \widetilde{y}_{11} & 0 & 0 \\ 0 & \widetilde{y}_{22} & 1 \\ 0 & 1 & \widetilde{y}_{22} \end{array} \right\| \end{array}\right\}, \tag{6.19}$$

where $\widetilde{x}_{11}+\widetilde{x}_{22}+\widetilde{x}_{33}=0$, $\widetilde{x}_{22}-\widetilde{x}_{33}=2$, $\widetilde{y}_{11}+2\widetilde{y}_{22}=0$ (so in this case there are 2 independent parameters determining the space structure by an invariant form),

$$\left.\begin{array}{c}\underline{\text{Kind III}}\\ \|C_{ab}\| = \left\| \begin{array}{cc} \widetilde{x} & \widetilde{y} \\ \widetilde{y} & -\widetilde{x} \end{array} \right\|, \\ \widetilde{x} = \left\| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\|, \qquad \widetilde{y} = \left\| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right\| \end{array}\right\}. \tag{6.20}$$

As was shown in §3, the diagonal components of the matrices x and y represent, respectively, the real and imaginary parts of stationary curvatures $\Lambda_i=\alpha_i+i\beta_i$ $(i=1,2,3)$ of the Riemann-Christoffel curvature tensor. Accordingly, we obtain stationary curvatures of Weyl's conformal curvature tensor. They are

$$\widetilde{\Lambda}_1 = \widetilde{x}_{11} + i\widetilde{y}_{11}\,, \quad \widetilde{\Lambda}_2 = \widetilde{x}_{22} + i\widetilde{y}_{22}\,, \quad \widetilde{\Lambda}_3 = \widetilde{x}_{33} + i\widetilde{y}_{33}\,. \tag{6.21}$$

As was mentioned above, the diagonal components of the matrix $\widetilde{y}$ coincide with the respective diagonal components of the matrix y.

Now, we write the formulae of the stationary curvatures while taking into account the obtained formulae of the components of Weyl's tensor $C_{\alpha\beta\gamma\delta}$, expressed through the chr.inv.-properties of the medium that fills the space. We obtain, for all three kinds of non-empty non-Einstein spaces, respectively

$$\left.\begin{array}{c}\underline{\text{Kind } \widetilde{\text{I}}}\\ \widetilde{\Lambda}_1 = -\frac{1}{c^2}\,X_{11} - \frac{\varkappa}{2c^2}\,U_{11} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{123}\\ \widetilde{\Lambda}_1 = -\frac{1}{c^2}\,X_{22} - \frac{\varkappa}{2c^2}\,U_{22} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{231}\\ \widetilde{\Lambda}_1 = -\frac{1}{c^2}\,X_{33} - \frac{\varkappa}{2c^2}\,U_{33} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{312}\end{array}\right\}, \tag{6.22}$$

$$\left.\begin{array}{c}\underline{\text{Sub-kind } \widetilde{\text{D}} \text{ of kind } \widetilde{\text{I}} \;\; (\widetilde{\Lambda}_2 = \widetilde{\Lambda}_3)}\\ X_{22} - X_{33} = \frac{\varkappa}{2}\,(U_{33} - U_{22})\\ Y_{231} = Y_{312}\end{array}\right\}, \tag{6.23}$$

$$\left.\begin{array}{c}\underline{\text{Sub-kind } \widetilde{\text{O}} \text{ of kind } \widetilde{\text{I}} \;\; (\widetilde{\Lambda}_1 = \widetilde{\Lambda}_2 = \widetilde{\Lambda}_3)}\\ X_{11} + \frac{\varkappa}{2}\,U_{11} = X_{22} + \frac{\varkappa}{2}\,U_{22} = X_{33} + \frac{\varkappa}{2}\,U_{33}\\ Y_{123} = Y_{231} = Y_{312} = 0\end{array}\right\}, \tag{6.24}$$

$$\left.\begin{array}{c}\underline{\text{Kind } \widetilde{\text{II}}}\\ \widetilde{\Lambda}_1 = -\frac{1}{c^2}\,X_{11} - \frac{\varkappa}{2c^2}\,U_{11} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{123}\\ \widetilde{\Lambda}_2 = -\frac{1}{c^2}\,X_{22} - 1 - \frac{\varkappa}{2c^2}\,U_{22} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{231} =\\ = -\frac{1}{c^2}\,X_{33} + 1 - \frac{\varkappa}{2c^2}\,U_{22} + \frac{\varkappa}{6}\left(\rho + \frac{2U}{c^2}\right) + \frac{i}{c}\,Y_{231}\end{array}\right\}, \tag{6.25}$$

$$\left.\begin{array}{c}\underline{\text{Sub-kind } \widetilde{\text{N}} \text{ of kind } \widetilde{\text{II}} \;\; (\widetilde{\Lambda}_1 = \widetilde{\Lambda}_2)}\\ X_{11} + \frac{\varkappa}{2}\,U_{11} = X_{22} + \frac{\varkappa}{2}\,U_{22} - c^2 = X_{33} + \frac{\varkappa}{2}\,U_{33} + c^2\\ Y_{123} = Y_{231} = Y_{312} = 0\end{array}\right\}, \tag{6.26}$$

$$\left.\begin{array}{c}\underline{\text{Kind }\widetilde{\text{III}}}\\ X_{11}+\frac{\varkappa}{2}U_{11}=X_{22}+\frac{\varkappa}{2}U_{22}=X_{33}+\frac{\varkappa}{2}U_{33}=0\\ Y_{123}=Y_{231}=Y_{312}=0\end{array}\right\}. \qquad (6.27)$$

As seen, in spaces of kind $\widetilde{\text{III}}$ all stationary curvatures are zero. However the aforementioned canonical representation of the matrix $\|C_{ab}\|$ of Weyl's tensor in an ortho-frame manifests that both matrices $\widetilde{x}$ and $\widetilde{y}$ are nonzero in any case, and this fact does not depend on the kind of matter that fills the space.

Finally, our consideration of the canonical forms of Weyl's conformal curvature tensor, and its stationary curvatures for non-empty non-Einstein spaces of all three kinds leads to the following conclusion:

> The presence of distributed matter (medium) in a non-Einstein space changes only the real parts of the stationary curvatures. The impossibility of gravitational waves and gravitational inertial waves, which is the condition $\widetilde{Y}_{ijk}=0$ (equality to zero of the dynamical observable component of Weyl's tensor), can only be realized in the non-empty spaces (gravitational field) of kind $\widetilde{\text{I}}$, where the stationary curvatures take real values. In non-empty non-Einstein spaces of the kinds other than kind $\widetilde{\text{I}}$, gravitational waves and gravitational inertial waves are possible.

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Behaviour of the EGR Persistent Vacuum Field Following the Lichnérowicz Matching Conditions

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Abstract: Recently, the author has proposed an extension of the General Theory of Relativity — the EGR theory, which allows for a persistent gravity-like field to exist as a homogeneous energy density background. In this paper, we demonstrate the continuity of this field with respect to the gravitational field of a massive body. To achieve this goal, we make use of the Lichnérowicz conjecture which formulates the conditions required to match a hyperbolic 4-metric characterized by a material-energy tensor, with a similar type of vacuum-solution metric. This is herein applied to a spherically symmetric class of the general relativistic solutions compatible with the Schwarzschild exterior metric. The EGR covariant derivatives of the metric are then only radial and time-dependent functions: the radial persistent field tensor component vanishes on a hypersurface separating the vacuum from the matter state. As a consequence, when this hypersurface is narrowed down to the size of a particle, it follows a non-Riemannian geodesic describing the trajectory of the particle whose mass slightly increased: this effect can be interpreted as the bare mass carrying its subsequent gravitational field.

Contents:

Introduction 72

Chapter 1 The Cauchy Problem in General Relativity

§1.1 Problem statement 73

§1.2 The exterior situation 73

§1.3 The interior situation 75

Chapter 2 Application to Spherically Symmetric Metrics

§2.1 The general solution 75

§2.2 The Schwarzschild metric (classical solution) 79

§2.3 The Schwarzschild metric (the EGR formulation) 81

Chapter 3 The Local Matching Conditions

§3.1 General definition 82

§3.2 Application to a natural basis 83

§3.3 Conditions for matching the EGR metrics 85

Discussion and concluding remarks 86

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Introduction. The problem of matching two Riemannian hyperbolic metrics in the sense of Lichnérowicz [1] can be stated as follows:

> Given a metric solution corresponding to a "normal material tensor" [2], we look for a hypersurface S where some "junction" conditions must be fulfilled to match a similar vacuum metric, so that some degrees of smoothness are not lost when approaching it form either side of the hypersurface.

This mathematical procedure is derived from the evolution of Einstein's equations, which necessarily involves the Cauchy problem.

In order to have an appropriate simple picture of the situation, we begin by regarding one of the "material metric" corresponding to a massive source, as generating a four-dimensional space-time "world tube". It is thus convenient to visualize the tube walls as a hypersurface S. Lichnérowicz admissible coordinates [3] can be introduced from either side of the hypersurface. Within the tube, the space metric satisfies the "material" Einstein equations. Outside the tube, the metric satisfies the source-free Einstein equations. The admissible coordinate conditions imply $G^4_a = 0$ for the Einstein tensor G_{ab} along with the time component $u^4 = 0$ of the unit vector u^a on the dividing hypersurface S. In this case, Lichnérowicz proved that the hypersurface S is generated by a congruence of time-like geodesics, since S is tangent to those lines and is thus itself time-like.

Let us imagine that the material tensor represents a massive particle, if the section of the tube is narrowing down to a particle size. In this case, we easily verify that such a particle would follow a time-like geodesic which is imposed by the field of the exterior metric.

Earlier on, guided by the equivalence principle whereby inertia is not locally distinguishable from gravitation, Einstein extended the special relativistic law of motion for a test particle to a gravitational field geodesic. On the other hand, the fundamental consequence of the matching conditions (which was later acknowledged by Einstein himself) results in the following: the geodesic principle is no longer a *postulate*, but a straightforward *consequence* of Einstein's equations.

In this paper, we will be primarily concerned with the discontinuity which the EGR persistent field [4] might undergo when switching from the source-free metric to the material (matter-filled) metric. To answer this question we are going to follow Lichnérowicz' program applied to a spherically symmetric class of Einstein's field solutions that are to match the Schwarzschild exterior metric. A particular importance is the assumption of a homogeneously distributed EGR field, which in this case results in an *extended Schwarzschild exterior solution* fully compatible with the standard Schwarzschild exterior solution (Riemannian geometry). Through this derivation, we are eventually led to reconsider the matter under the form of a modified density.

Chapter 1. The Cauchy Problem in General Relativity

§1.1. Problem statement. As is well known, Einstein's equations are non-linear. The gravitational fields corresponded to the equations, even when singled out, define the space-time over which they propagate. As a result, the solution of the equations can be found to be unique up to a diffeomorphism, and hence one is forced to introduce a fixed background or hypersurface S onto which a set of initial data are given. From these Cauchy data, it is thus possible to predict and study the further evolution of Einstein's equations in the neighbourhood of S.

The Cauchy problem in General Relativity was pioneered by Darmois and Lichnérowicz [5], then extensively studied in [6]. We restrict this topic to local considerations of the problem. For a full treatment of the global aspect of the Cauchy problem in General Relativity, see for instance Choquet-Bruhat and Geroch [7, 8], and others [9–12]. From a strict mathematical point of view, the Cauchy problem can be formulated as follows [13]:

> Let S be a given three-dimensional manifold and a set of n initial data on it. We look for a four-dimensional Lorentzian manifold (M, g) and an embedding $f : \mathrm{S} \to \mathrm{M}$ such that the metric $\mathrm{g} = g_{ab}\, dx^a \otimes dx^b$ satisfies Einstein's equations and the initial conditions on $f(\mathrm{S})$, and that $f(\mathrm{S})$ constitutes a Cauchy hypersurface for the manifold (M, g).

§1.2. The exterior situation. Following Lichnérowicz, we assume that components of the metric tensor g_{ab} (as well as their first derivatives) should be smooth and continuous on a given hypersurface S. In the neighbourhood of S of any event, the potentials g_{ab} satisfy the source-free Einstein equations

$$G_{ab} = 0\,, \qquad a, b = 1, 2, 3, 4, \tag{1.1}$$

where the right-hand side can include the cosmological term.

We consider a space-like hypersurface $x^4 = 0$: therein g_{ab} and their first derivatives $\partial_4\, g_{ab}$ are thus defined as the set of n initial data.

From the contracted Bianchi identities

$$G^{a4}_{,4} = -\,G^{a\sigma}_{,\sigma} - \left\{{a \atop bc}\right\} G^{cb} - \left\{{b \atop bc}\right\} G^{ac} \tag{1.2}$$

we see that the right hand side contains at most two differentiations with respect to time and so it must be the case for the left-hand side. Therefore,

$$G^{a4} = 0 \tag{1.3}$$

contains only first derivatives of the metric tensor with respect to time.

The second-order derivative $\partial_{44}\, g_{ab}$ cannot be determined by the field equations. Hence, no information can be extracted about the time evolution from the four equations (1.3).

These equations are regarded as the *constraint Einstein equations* for the set of n initial data, i.e. for g_{ab} and $\partial_4\, g_{ab}$. If they are satisfied by the initial data, there exists a solution of the Cauchy problem for the field equations $G_{ab} = 0$ in the neighbourhood of S.

So, we are left with 6 dynamical field equations

$$G_{\alpha\beta} = 0\,, \qquad \alpha, \beta = 1, 2, 3. \tag{1.4}$$

For the second-order derivatives $\partial_{44}\, g_{ab}$ (they are 10), we have a four-fold ambiguity which can be removed by imposing four conditions (known as the *harmonicity conditions*) on the metric tensor g_{ab}.

Explicitly, these conditions are

$$F^b = \frac{\partial \mathcal{G}^{ab}}{\partial x^a} = 0\,, \tag{1.5}$$

where $\mathcal{G}^{ab} = \sqrt{-g}\, g^{ab}$ is the metric tensor density. With this choice of harmonic coordinates, the Einstein tensor G_{ab} can be written as

$$G^{ab} = \left(G^{ab}\right)_{\rm harm} + A^{ab} \tag{1.6}$$

with

$$\left.\begin{aligned} A^{ab} &= \frac{1}{2}\left(g^{ac}\,\partial_c F^b + g^{bc}\,\partial_c F^a\right) \\ \left(G^{ab}\right)_{\rm harm} &= \frac{g^{ik}}{2\sqrt{-g}}\,\frac{\partial^2 \mathcal{G}^{ab}}{\partial x^i \partial x^k} + H^{ab} \end{aligned}\right\}, \tag{1.7}$$

where H_{ab} depends on the potentials and their first derivatives.

Hence, we can solve the *reduced Einstein equations*

$$\left(G^{ab}\right)_{\rm harm} = 0\,. \tag{1.8}$$

The solutions of the initial problem should satisfy the constraint equations (1.3) at any later time.

Consider the conservation equations

$$\nabla_a\, G^a_b = 0\,, \tag{1.9}$$

that is

$$\nabla_4 G^4_b + \nabla_\alpha G^\alpha_b = 0\,. \tag{1.10}$$

The constraints (1.3) are imposed so that G^{ab} vanishes everywhere along with $G^{\alpha\beta}$. It can also be shown that, taking into account (1.10) on S where (1.3) is satisfied, the constraint equations are also satisfied in the neighbourhood of S.

Therefore the equations (1.3) propagate, and the Einstein equations are said to be in *involution* (not evolution), in the sense of Cartan.

§1.3. Interior situation. Here the problem is somewhat more complex. Put it simply, the field equations are a part of the system

$$\left.\begin{aligned} G^{ab} &= \varkappa T^{ab} \\ \nabla_a T^{ab} &= 0 \end{aligned}\right\}, \tag{1.11}$$

which is also in involution in the sense of Cartan. On the hypersurface S ($x^4 = const$), we choose initial data satisfying the four conditions

$$G^4_a = \varkappa T^4_a \tag{1.12}$$

for $x^4 = 0$. Inspection shows that the Cauchy problem has a solution in the neighbourhood of S, provided that the data are sufficiently differentiable in the case of a massive tensor.

Chapter 2. Application to Spherically Symmetric Metrics

§2.1. The general solution. We begin by redefining a spherically symmetric Lorentzian manifold (M, g) as a manifold admitting the group SO(3) as an isometric group, in such a way that the group orbits are two-dimensional space-like surfaces.

The group orbits are necessarily surfaces of constant positive curvature. Thus, it is always possible to introduce coordinates such that the metric has the regular form

$$ds^2 = e^{2a(T,R)}dT^2 - e^{2b(T,R)}dR^2 + e^{2c(T,R)}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.1}$$

According to the EGR theory, 1) we keep the spherical symmetry and maintain the normalization of R so that a circle has the circumference $2\pi R$; 2) we make the legitimate assumption that the EGR covariant metric tensor variations only apply to T and R.

The general form of the EGR metric

$$(ds^2)_{\text{EGR}} = ds^2 + dJ \tag{2.2}$$

has been postulated [4], where the linear form $dJ = f(J_a)\,dx^a$ depends on the covariant derivative of the metric tensor

$$\mathrm{D}_a\, g_{bc} = \frac{1}{3}\left(J_c\, g_{ab} + J_b\, g_{ac} - J_a\, g_{bc}\right). \tag{2.3}$$

With our second assumption, we write the spherically symmetric EGR metric as

$$(ds^2)_{\mathrm{EGR}} = ds^2 + (J_T\, dT - J_R\, dR)\,. \tag{2.4}$$

A quick comparison with (2.1) readily leads to $dT\,(e^{2a(T,R)}\,dT + J_T)$, which we write in the form $dT^2\,(e^{2\mathcal{A}})$. In the same way, $dR^2\,(e^{-2\mathcal{B}})$. Finally, we have the modified coefficients

$$\left.\begin{aligned} \mathcal{A} &= a + \text{correction}\,(R,T)\\ \mathcal{B} &= b + \text{correction}\,(R,T)\\ \mathcal{C} &= c + \text{correction}\,(R,T) \end{aligned}\right\}, \tag{2.5}$$

thus we write the EGR spherical metric in the form

$$(ds^2)_{\mathrm{EGR}} = e^{2\mathcal{A}(R,T)}dT^2 - e^{-2\mathcal{B}(R,T)}dR^2 - \\ - e^{2\mathcal{C}(R,T)}\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big). \tag{2.6}$$

Using Cartan's calculus, we will be able to obtain formulae for the EGR Ricci tensor and the EGR Einstein tensor.

First, we re-write the metric (2.6) with the Pfaffian forms

$$(ds^2)_{\mathrm{EGR}} = (\omega^4)^2 - (\omega^a)^2, \tag{2.7}$$

where the local basis Pfaffian forms are given by

$$\omega^4 = e^{\mathcal{A}dT}dT, \quad \omega^1 = e^{\mathcal{B}}dR, \quad \omega^2 = e^{\mathcal{C}}d\theta, \quad \omega^3 = e^{\mathcal{C}}\sin\theta\, d\varphi. \tag{2.8}$$

Now, we need the *connection forms*, which will be obtained from the first Cartan structure equations

$$d\omega = -\,\omega^a_b \wedge \omega^b. \tag{2.9}$$

Determining first the exterior derivatives

$$\left.\begin{aligned} d\omega^4 &= \mathcal{A}' e^{-\mathcal{B}}\omega^1\wedge\omega^4\\ d\omega^1 &= \dot{\mathcal{B}}\, e^{-\mathcal{A}}\omega^4\wedge\omega^1\\ d\omega^2 &= \dot{\mathcal{C}}\, e^{-\mathcal{A}}\omega^4\wedge\omega^2 + \dot{\mathcal{C}}\, e^{-\mathcal{B}}\omega^1\wedge\omega^2\\ d\omega^3 &= \dot{\mathcal{C}}\, e^{-\mathcal{A}}\omega^4\wedge\omega^3 + \dot{\mathcal{C}}\, e^{-\mathcal{B}}\omega^1\wedge\omega^3 + \frac{1}{R}\cot\theta\left(\omega^2\wedge\omega^3\right) \end{aligned}\right\}, \tag{2.10}$$

where $\mathcal{A}' = \frac{\partial \mathcal{A}}{\partial R}$ and $\dot{\mathcal{B}} = \frac{\partial \mathcal{B}}{\partial T}$, then using of (2.9), we find

$$\left.\begin{aligned} \omega_1^4 &= \omega_4^1 = \mathcal{A}' e^{-\mathcal{B}} \omega^4 + \dot{\mathcal{B}} e^{-\mathcal{A}}\, \omega^1 \\ \omega_2^4 &= \omega_4^2 = \dot{\mathcal{C}}\, e^{-\mathcal{A}}\, \omega^2 \\ \omega_3^4 &= \omega_4^3 = e^{-\mathcal{A}}\, \dot{\mathcal{C}}\, \omega^3 \\ \omega_1^2 &= -\,\omega_2^1 = \mathcal{C}' \\ \omega_1^3 &= -\,\omega_3^1 = \mathcal{C}'\, e^{-\mathcal{B}}\, \omega^3 \\ \omega_2^3 &= \omega_3^2 = -\frac{1}{R}\, \cot\theta\, \omega^3 \end{aligned}\right\} . \tag{2.11}$$

The ansatz (2.11) satisfies

$$\omega_{ab} + \omega_{ba} = 0\,, \tag{2.12}$$

since the basis ω_a is chosen to be orthonormal.

From the second structure equation

$$\Omega_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c\,, \tag{2.13}$$

we obtain the EGR curvature forms

$$\left.\begin{aligned} \Omega_1^4 &= E\,\omega^4 \wedge \omega^1 \\ \Omega_2^4 &= F\,\omega^4 \wedge \omega^2 + H\omega^1 \wedge \omega^2 \\ \Omega_3^4 &= F\,\omega^4 \wedge \omega^3 + H\omega^1 \wedge \omega^3 \\ \Omega_2^1 &= I\,\omega^1 \wedge \omega^2 - H\omega^4 \wedge \omega^2 \\ \Omega_3^1 &= I\,\omega^1 \wedge \omega^3 - H\omega^4 \wedge \omega^3 \\ \Omega_3^2 &= D\,\omega^2 \wedge \omega^3 \end{aligned}\right\} , \tag{2.14}$$

where we use the short denotations

$$\left.\begin{aligned} E &= e^{-2\mathcal{A}}\big(\ddot{\mathcal{B}} + \dot{\mathcal{B}}^2 - \dot{\mathcal{B}}\dot{\mathcal{A}}\big) - e^{-2\mathcal{B}}\big(\mathcal{A}'' + \mathcal{A}'^2 - \mathcal{A}'\mathcal{B}'\big) \\ F &= e^{-(\mathcal{A}+\mathcal{B})}\big(\dot{\mathcal{C}}' + \dot{\mathcal{C}}\mathcal{C}' - \dot{\mathcal{C}}\mathcal{A}' - \dot{\mathcal{B}}\mathcal{C}'\big) \\ H &= e^{-2\mathcal{A}}\big(\ddot{\mathcal{C}} + \dot{\mathcal{C}}^2 - \dot{\mathcal{C}}\dot{\mathcal{A}}\big) - e^{-2\mathcal{B}}\mathcal{A}'\mathcal{C}' \\ D &= e^{-2\mathcal{A}}\dot{\mathcal{C}}^2 - e^{-\mathcal{B}}\mathcal{C}'^2 + e^{-2\mathcal{C}} \\ I &= e^{-2\mathcal{A}}\dot{\mathcal{C}}\dot{\mathcal{B}} - e^{-2\mathcal{B}}\big(\mathcal{C}'' + \mathcal{C}'^2 - \mathcal{C}'\mathcal{B}'\big) \end{aligned}\right\} . \tag{2.15}$$

Hence, we can infer the needed diagonal components of the EGR Ricci tensor $(R_{ab})_{\text{EGR}}$. We obtain

$$\left.\begin{aligned} (R_{44})_{\text{EGR}} &= -E - 2F \\ (R_{11})_{\text{EGR}} &= E + 2I \\ (R_{22})_{\text{EGR}} &= (R_{33})_{\text{EGR}} = E + D + I \end{aligned}\right\}, \tag{2.16}$$

while the curvature scalar is given by

$$(R)_{\text{EGR}} = -\,2\,(E + I) - 4\,(F + I)\,. \tag{2.17}$$

We can now calculate the useful components of the EGR Einstein tensor [4], which is defined as follows

$$(G_{ab})_{\text{EGR}} = (R_{ab})_{\text{EGR}} - \frac{1}{2}\left[g_{ab}\,(R)_{\text{EGR}} - \frac{2}{3}\,J_{ab}\right]. \tag{2.18}$$

In our particular case, the diagonal components reduce this tensor to the Riemannian form

$$G_{ab} = R_{ab} - \frac{1}{2}\,g_{ab}\,R\,, \tag{2.19}$$

so that the diagonal components of the EGR Einstein tensor are

$$\left.\begin{aligned} (G^4_4)_{\text{EGR}} &= (G_{44})_{\text{EGR}} = D + 2I \\ (G^1_1)_{\text{EGR}} &= (G_{11})_{\text{EGR}} = 2F + D \\ (G_{22})_{\text{EGR}} &= (G_{33})_{\text{EGR}} = E + I + F \end{aligned}\right\}. \tag{2.20}$$

According to the EGR theory [4], for the *interior metric*, and assuming for the EGR unit 4-velocity that $(u_a u^a)_{\text{EGR}} = 1$, these components are associated with the material tensor and the persistent field

$$(G^4_4)_{\text{EGR}} = \varkappa\left[\,T^4_4 + (t^4_4)_{\text{EGR}}\right] = \varkappa\left[\,\rho + (t^4_4)_{\text{EGR}}\right] = \varkappa\rho^*, \tag{2.21}$$

$$(G^1_1)_{\text{EGR}} = \varkappa\,(t^1_1)_{\text{EGR}}\,, \tag{2.22}$$

and $(G_{22})_{\text{EGR}} = \varkappa\,(t_{22})_{\text{EGR}}$, $(G_{33})_{\text{EGR}} = \varkappa\,(t_{33})_{\text{EGR}}$, thus

$$(t_{22})_{\text{EGR}} = (t_{33})_{\text{EGR}}\,, \tag{2.23}$$

where ρ^* stands for the modified material density, which was already introduced by the EGR theory.

Our main task will be to show that the external (radial) component of the EGR persistent field is vanishing at the contact of the spherical mass (source of the field), thus ensuring the continuity with the modified massive density. To be more specific, we expect to see that the vacuum EGR persistent field will actually vanish when approaching asymptotically a region around the bare mass. Such a global region represents the modified massive quantity, i.e. the bare mass carries its subsequent gravitational field.

§2.2. The Schwarzschild metric (classical solution). We first consider the classical Schwarzschild solution, which is obtained in the framework of Riemannian geometry. Then we represent it according to the EGR theory.

As is known, the Schwarzschild metric in the spherical coordinates has the form

$$ds^2 = e^{2a(r)}\, dt^2 - e^{2b(r)}\, dr^2 - r^2 \left(d\zeta^2 + \sin^2\!\zeta\, d\varphi^2\right). \tag{2.24}$$

We rewrite this linear element with the Pfaffian forms

$$ds^2 = (\theta^4)^2 - (\theta^a)^2, \tag{2.25}$$

where we have chosen

$$\theta^4 = e^a dt\,, \quad \theta^1 = e^b\, dr\,, \quad \theta^2 = r\, d\zeta\,, \quad \theta^3 = r\,\sin\zeta\, d\varphi\,. \tag{2.26}$$

Exterior differentiation of these results immediately in

$$\left.\begin{aligned} d\theta^4 &= a' e^a dr \wedge dt \\ d\theta^1 &= 0 \\ d\theta^2 &= dr \wedge d\zeta \\ d\theta^3 &= \sin\zeta\, dr \wedge d\varphi + r\cos\zeta\, d\zeta \wedge d\varphi \end{aligned}\right\}. \tag{2.27}$$

Comparison with the first structure equation leads to the following expressions for the connection forms

$$\left.\begin{aligned} \omega_1^4 &= \omega_4^1 = a' e^{-b}\, \theta^4 \\ \omega_1^2 &= \omega_2^1 = \frac{1}{r}\, e^{-b}\, \theta^2 \\ \omega_1^3 &= -\omega_3^1 = \frac{1}{r}\, e^{-b}\, \theta^3 \\ \omega_2^3 &= -\omega_3^2 = \frac{1}{r}\, (\cot\zeta)\, \theta^3 \\ \omega_2^4 &= \omega_4^2 = \omega_3^4 = \omega_4^3 = 0 \end{aligned}\right\}. \tag{2.28}$$

From the second structure equation, we obtain the curvature forms

$$\left.\begin{aligned} \Omega^4_1 &= e^{-2b}\big(a'b' - a'' - a'a'\big)\,\theta^4\wedge\theta^1 \\ \Omega^4_2 &= -\frac{a'e^{-b}}{r}\,\big(\theta^4\wedge\theta^2\big) \\ \Omega^4_3 &= -\frac{a'e^{-2b}}{r}\,\big(\theta^4\wedge\theta^3\big) \\ \Omega^1_2 &= \frac{b'e^{-2b}}{r}\,\big(\theta^1\wedge\theta^2\big) \\ \Omega^1_3 &= \frac{b'e^{-2b}}{r}\,\big(\theta^1\wedge\theta^3\big) \\ \Omega^2_3 &= \frac{1-e^{-2b}}{r^2}\,\big(\theta^2\wedge\theta^3\big) \end{aligned}\right\}. \tag{2.29}$$

For the Einstein tensor, we obtain the useful mixed diagonal components

$$G^4_4 = \frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right), \tag{2.30}$$

$$G^1_1 = \frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right), \tag{2.31}$$

$$G^2_2 = G^3_3 = -e^{-2b}\left(a'^2 - a'b' + a'' + \frac{a'-b'}{r}\right). \tag{2.32}$$

The vacuum solutions are then given by

$$G^4_4 + G^1_1 = 0\,, \tag{2.33}$$

which imply that $a' + b'$, and hence $a + b = 0$, since a, b approach zero asymptotically such that the Schwarzschild metric becomes asymptotically flat, i.e. $a = -b$.

Integrating (2.30), we obtain

$$e^{2a} = e^{-2b} = 1 - \frac{m}{r}\,. \tag{2.34}$$

The constant m is determined as follows: at large distances we must have the Newtonian limit

$$g^{44} \approx 1 + 2U, \tag{2.35}$$

where $U = -\frac{\mathfrak{G}M}{r}$ is the classical gravitational potential, where M is the mass producing the field. Hence, $m = \mathfrak{G}M$ (we have assumed $c = 1$).

Once $\left(1-\frac{2m}{r}\right)$ has been substituted into the curvature forms (2.29), we find, for the curvature tensor components,

$$R_{4141}=-R_{2323}=2L\,,\quad R_{1212}=R_{1313}=R_{4242}=-R_{4343}=L\,, \qquad (2.36)$$

where

$$L=\frac{m}{r^3}\,. \qquad (2.37)$$

§2.3. The Schwarzschild metric (the EGR formulation). Following the same procedure as in §2.2, we write the extended Schwarzschild solution

$$(ds^2)_{\text{EGR}}=e^{2A(r,t)}\,dt^2-e^{2B(r,t)}\,dr^2-r^2\left(d\zeta^2+\sin^2\!\zeta\,d\varphi^2\right), \qquad (2.38)$$

where the coefficients A and B are formulated as

$$A=a+\text{correction }(r,t)\,,\quad B=b+\text{correction }(r,t)\,. \qquad (2.39)$$

In Riemannian geometry, the Schwarzschild metric is obtained as a vacuum solution. According to the EGR theory, there is not source-free solution: the field equations are characterized by a persistent field t_{ab}. Therefore, applying the "vacuum" formulae (2.30) and (2.31) of the classical Schwarzschild solution, we obtain

$$(G^4_4)_{\text{EGR}}=\varkappa\,t^4_4\,,\quad (G^1_1)_{\text{EGR}}=\varkappa\,t^1_1\,, \qquad (2.40)$$

thus we have

$$(G^4_4)_{\text{EGR}}+(G^1_1)_{\text{EGR}}=\varkappa\left(t^4_4+t^1_1\right). \qquad (2.41)$$

According to the EGR theory, the persistent field is assumed to be homogeneously distributed as a background energy density. Under the assumption of spherical symmetry we thus need only the components (2.40), so that we have

$$t^4_4=-\,t^1_1\,. \qquad (2.42)$$

The obtained EGR formula (2.41) is similar to that according to Riemannian geometry (2.32). Therefore, we have the most important result which formulates as:

> According to the aforementioned (mixed) diagonal conditions, the classical Schwarzschild exterior solution is equivalent to the EGR Schwarzschild metric.

This circumstance enables us to set forth

$$e^{2A}=e^{-2B}=1-\frac{m^*}{r}\,, \qquad (2.43)$$

where m^* is a *modified mass* we have introduced through the relation

$$m^* = m + \text{correction}. \tag{2.44}$$

We note here the very consistency with our previous result (2.21), where we have been able to determine a modified density ρ^*. We thus extrapolate (2.37) as

$$(L)_{\text{EGR}} = \frac{m^*}{r^3}\,. \tag{2.45}$$

The corresponding EGR curvature forms are

$$\left.\begin{aligned}
(\Omega^4_1)_{\text{EGR}} &= 2\,(L)_{\text{EGR}}\left(\theta^{4*}\wedge\theta^{1*}\right)\\
(\Omega^4_2)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{4*}\wedge\theta^{2*}\right)\\
(\Omega^4_3)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{4*}\wedge\theta^{3*}\right)\\
(\Omega^2_3)_{\text{EGR}} &= 2\,(L)_{\text{EGR}}\left(\theta^{2*}\wedge\theta^{3*}\right)\\
(\Omega^1_3)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{3*}\wedge\theta^{1*}\right)\\
(\Omega^1_2)_{\text{EGR}} &= -\,(L)_{\text{EGR}}\left(\theta^{1*}\wedge\theta^{2*}\right)
\end{aligned}\right\},\tag{2.46}$$

where θ^{a*} are the *EGR Pfaffian forms* which are determined by the EGR coefficients of the metric (2.38).

Chapter 3. The Local Matching Conditions

§3.1. General definition. In a space-time manifold (M, g), where matter generates a world-tube limited by a hypersurface S, we are in the presence of an interior metric satisfying the massive field equations, and an exterior metric satisfying the source-free Einstein equations. From either side, g_{ab} are defined and smoothly and continuous in each open sub-domain. The purpose of the current work is to analyse the continuous properties required for the metrics when approaching and crossing S. To start with, we indicate the matching conditions as was first stated by Lichnérowicz:

> Given $x \in S$, there exists a frame of admissible coordinates whose domain includes x, and the potentials g_{ab} (related to this frame) as well as their first derivatives be continuous when crossing S.

Anticipating on the final proof result, Lichnérowicz also showed that the matching conditions requires for S to be a time-like hypersurface.

Let a Riemannian metric

$$ds_1^2 = g_{ab}\,dx^a dx^b \tag{3.1}$$

be defined on an open subset $\mathrm{O}_1 \subset (\mathrm{M}, \mathrm{g})$ (a four-dimensional manifold). Denote the respective metric tensor and Riemannian connection as $\mathrm{g}_1(x)$ and $\{\ \}_1(x)$, with $x \in \mathrm{O}_1$. Consider another metric

$$ds_2^2 = g_{c'd'}\, dx^{c'} dx^{d'} \tag{3.2}$$

defined on O_2 which is connected to (M, g), with $\mathrm{g}_2(x')$ and $\{\ \}_2(x')$ (we mean here that $x' \in \mathrm{O}_2$).

Provided that ds_1^2 and ds_2^2 are attributed to the same hyperbolic type (and having the same signature), they can be matched in the sense of Lichnérowicz, if there exists:

1) Functions

$$x^{e'} = x^{e'}(x^a)\,, \tag{3.3}$$

whose non-vanishing Jacobian $J_a^{c'} = \frac{\partial x^{c'}}{\partial x^a}$ satisfies $J_a^{c'} J_{c'}^b = \delta_a^b$;

2) A hypersurface S represented by a local equation $f(x^a) = 0$ on which holds

$$g_{ab} = J_a^{i'} J_b^{k'} g_{i'k'}\,, \tag{3.4}$$

$$\left\{ {c \atop ab} \right\} = J_{d'}^c J_a^{i'} J_b^{k'} \left\{ {d' \atop i'k'} \right\} + J_{d'}^c\, \partial_a J_b^{d'}\,. \tag{3.5}$$

§3.2. Application to a natural basis. In view of applying our next program for the matching conditions, it suffices to adopt the approximated Minkowskian forms of the metrics (3.1) and (3.2)

$$ds_1^2 = \eta_{ab}\, \omega^a \wedge \omega^b, \tag{3.6}$$

$$ds_2^2 = \eta_{e'l'}\, \theta^{e'} \wedge \theta^{l'}, \tag{3.7}$$

where the Pfaffian forms are defined by

$$\omega^a = A_b^a\, dx^b, \qquad \theta^{e'} = B_{l'}^{e'}\, dx^{l'}. \tag{3.8}$$

The change of variables (3.3) performed on $\theta^{e'}$ yields

$$\theta^{e'} = \mathcal{L}_a^{e'} \omega^a, \tag{3.9}$$

where

$$\mathcal{L}_a^{e'} = B_{l'}^{e'} \mathcal{L}_b^{l'} (A^{-1})_a^b\,. \tag{3.10}$$

It can be shown that the conditions (3.5) are equivalent to

$$\omega_b^a = \mathcal{L}_{e'}^a\, \theta_{l'}^{e'} \mathcal{L}_b^{l'} + \mathcal{L}_{k'}^a\, d\mathcal{L}_b^{k'}\,. \tag{3.11}$$

Since we consider a Lorentzian manifold (M, g) with $n=4$, the matrix $\mathcal{L}$ is an element of the special Lorentz group which is reduced into the boost

$$\mathcal{L}_{a}^{e'} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}\,. \tag{3.12}$$

The Lorentz invariance is also obtained by setting $\tanh\chi = v$, in which case we have for $\mathcal{L}$ the following form

$$\mathcal{L}_{e'}^{a} = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.13}$$

$$\mathcal{L}_{a}^{e'} = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.14}$$

where the parameter χ will be defined later on.

Remarkably, we check in passing that the components of the Riemann tensor in the classical Schwarzschild metric (2.24) are unchanged with respect to a radial co-moving reference frame ($u^{\alpha}=0$, $u_4 u^4=1$)

$$\begin{aligned} R'_{4141} &= \mathcal{L}_{4'}^{a}\,\mathcal{L}_{1'}^{b}\,\mathcal{L}_{4'}^{c}\,\mathcal{L}_{1'}^{d}\,R_{abcd} = \\ &= R_{4141} = \left(\cosh^4\chi - 2\cosh^2\chi\,\sinh^2\chi + \sinh^4\chi\right) R_{4141}\,, \end{aligned} \tag{3.15}$$

since $\cosh^2\chi - \sinh^2\chi = 1$. In a similar manner, inspection shows the invariance of the other components in this particular frame of reference.

Let us now set

$$\mathcal{Q}_{b}^{a} = \omega_{b}^{a} - \mathcal{L}_{e'}^{a}\,\theta_{l'}^{e'}\,\mathcal{L}_{b}^{l'} - \mathcal{L}_{k'}^{a}\,d\mathcal{L}_{b}^{k'}\,. \tag{3.16}$$

According to (3.11), the form $\mathcal{Q}_{b}^{a}$ should be zero on the hypersurface S. Therefore the surface element $d\mathcal{Q}_{b}^{a}$ must satisfy

$$d\mathcal{Q}_{b}^{a} \wedge df = 0\,, \tag{3.17}$$

where df is the normal to S.

We are going to find the matching relation between the considered two metrics referred to the same basis ω^{a}.

To this effect, we first notice that the curvature form related to the metrics ds_1^2 and ds_2^2 is expressed, respectively, by

$$\Omega_b^a = \frac{1}{2}\, R^{a\cdots}_{\cdot bki}\, \omega^k \wedge \omega^i, \tag{3.18}$$

$$\Omega_{l'}^{e'} = \frac{1}{2}\, R^{e'\cdots}_{\cdot\, l'c'd'}\, \omega^{c'} \wedge \omega^{d'}. \tag{3.19}$$

Hence,

$$\mathcal{L}_{e'}^{a}\, \Omega_{l'}^{e'} \mathcal{L}_b^{l'} = \frac{1}{2}\, \mathcal{L}_{e'}^{a}\, R^{e'\cdots}_{\cdot\, l'c'd'}\, \mathcal{L}_b^{l'} \mathcal{L}_k^{c'} \mathcal{L}_i^{d'}\, \omega^k \wedge \omega^i \tag{3.20}$$

and $d\mathcal{Q}_b^a \wedge df$ is now written

$$\left(\Omega_b^a - \mathcal{L}_{e'}^{a}\, \Omega_{l'}^{e'} \mathcal{L}_b^{l'}\right) \wedge df = 0 \tag{3.21}$$

on the hypersurface S.

In particular, the latter equation can be written in terms of the Einstein tensor G_{ab} as

$$\left(G_b^a - \mathcal{L}_{e'}^{a}\, G_{l'}^{e'} \mathcal{L}_b^{l'}\right) \partial_a f = 0 \tag{3.22}$$

on the hypersurface S (this has been formulated, in another form than 3.22, by Lichnérowicz [5, p. 62]).

§3.3. Conditions for matching the EGR metrics. The curvature form is here given by

$$(\Omega_b^a)_{\text{EGR}} = \frac{1}{2}\, (R^{a\cdots}_{\cdot bki})_{\text{EGR}}\, \omega^k \wedge \omega^i, \tag{3.23}$$

where the EGR curvature tensor has the form

$$(R_{abki})_{\text{EGR}} = R_{abki} + B_{abki}\,, \tag{3.24}$$

where

$$B_{abki} = B_{abki}(J_{mn})\,, \quad J_{mn} = \partial_m J_n - \partial_n J_m\,, \tag{3.25}$$

and the Pfaffian forms ω_a are adapted accordingly.

We then denote the EGR Schwarzschild solutions as $(ds_1^2)_{\text{EGR}}$ and $(ds_2^2)_{\text{EGR}}$. The spherical symmetry suggests us to set

$$\theta = \zeta\,, \qquad \phi = \varphi\,, \qquad t = t\,(T)\,. \tag{3.26}$$

In order to investigate the possible consequences of matching the EGR metrics $(ds_1^2)_{\text{EGR}}$ and $(ds_2^2)_{\text{EGR}}$, we compute the exact components of $\mathcal{Q}^a_b$ with the help of (2.29) and (2.46), where the EGR Pfaffian forms of ds^2_2 are implicitly denoted by θ^{a*}. We eventually obtain

$$\left.\begin{aligned} \mathcal{Q}^4_1 &= (E-2L)\,\omega^4\wedge\omega^1 \\ \mathcal{Q}^4_2 &= (F+L)\,\omega^4\wedge\omega^2 \\ \mathcal{Q}^4_3 &= (F+L)\,\omega^4\wedge\omega^3 \\ \mathcal{Q}^2_3 &= (D-2L)\,\omega^2\wedge\omega^3 \\ \mathcal{Q}^3_1 &= (I+L)\,\omega^3\wedge\omega^1 \\ \mathcal{Q}^1_2 &= (I+L)\,\omega^1\wedge\omega^2 \end{aligned}\right\}. \tag{3.27}$$

A short inspection shows that fulfilling the condition (3.17), implies that we must set

$$df \propto \omega^1, \tag{3.28}$$

$$D-2L=0\,, \qquad F+L=0 \tag{3.29}$$

on S. That is the hypersurface S is *time-like* as it should be.

Indeed, had we chosen $df=d\omega^4$, we would then have been left with vanishing conditions involving the terms E and $I\neq 0$, whose coefficient $\mathcal{B}$ is time-dependent, and therefore contradicting the nature of the hypersurface S which would be space-like in this case.

From (3.28) and (3.29), we have

$$2F+D=0 \tag{3.30}$$

on S, which, taking (2.40) into account, yields the fundamental result

$$(t^1_1)_{\text{EGR}}=0\,. \tag{3.31}$$

The radial component of the EGR persistent field tensor vanishes on the time-like hypersurface S.

Ultimately, as is easy to show, the EGR coefficients of (2.5) and (2.39) define the parameter χ of (3.13) and (3.14) so that

$$\sinh\chi = -\,\dot{C}\,e^{B+C-\mathcal{A}}, \quad \cosh\chi = C'e^{B+C-\mathcal{B}}. \tag{3.32}$$

Discussion and concluding remarks. Under the above symmetry assumptions, the radial component is only the "dynamical" component, which is of importance here.

Therefore, we clearly see that, provided that the hypersurface S strictly divides the exterior EGR Schwarzschild solution from the EGR spherically symmetric interior metric, there exists a physical continuity between the exterior EGR persistent field tensor and the interior modified material tensor.

In Riemannian geometry, an *interior* spherically symmetric class of solutions of Einstein's equations corresponds to a normal material-energy tensor, i.e., to that of the generic form

$$T_{ab} = \rho\, u_a u_b - \Pi_{ab} \tag{3.33}$$

compatible with the spherical symmetry.

It was shown [14, 15] that under the assumption of spherical symmetry ($u^{\alpha} = 0,\ u_k u^k = 1$), all such *interior* solutions can be matched with the Schwarzschild exterior solution, provided that the radial pressure component $\Pi_1^1 = p$ vanishes on the time-like hypersurface S. This purely theoretical result has not any physical grounds.

On the contrary, the EGR theory provides here a much better interpretation: the continuity of the EGR persistent field presents indeed a physical consistency with the Lichnérowicz conjecture imposed as metric-matching conditions, which is a direct consequence of the Cauchy problem.

Following this pattern applied to two spherically symmetric models, it has indeed been rigorously shown that the EGR persistent field which pre-exists in the EGR "no-mass" metric, vanishes on the *contact separation* S between another metric containing a material source.

Reverting to the aforementioned picture where the "S-tube" section is considered as narrowing down to a particle's size, we can extend this proof by stating that the resulting principle of geodesics, still holds in the EGR theory for a neutral particle.

The essential difference lies in that the time-like geodesic is derived from the non-Riemannian EGR connection. As a result, the material source behaves as if it was modified by the "absorbed EGR field" presented in the matter.

As a matter of fact that a body's mass is not affected by the absorption of the EGR persistent field, but rather, the mass is now considered together with its own gravitational field, which has so far implicitly been described by an energy-momentum pseudo-tensor.

The EGR theory allows for an explicit description of a massive particle accompanied by its gravitational field, thus forming a single dynamical entity. If one still adopts the Riemannian picture, the "bare" proper mass of the particle is seen as being subjected to the influence of an environmental hidden medium that causes this mass to "fluctuate", according to de Broglie's Double Solution Theory [16]. Now, we clearly see that the random fluctuations are the manifestation of the particle's gravitational field, which is linked to the surrounding EGR field.

In conclusion, it should be noted that of importance is a pertinent analysis about the diagonal Gauss coordinates adopted in the framework of the admissible Lichnérowicz coordinate conditions, and related to the matching conditions applied to the Schwarzschild metric [17].

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Geodesics and Finslerian Equations in the EGR Theory

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Abstract: In the framework of the EGR theory, introduced recently by the author as a non-Riemannian extension of the General Theory of Relativity, the geodesic equations for a free neutral particle have the same form as those in Riemannian geometry except that they describe the particle's motion together with its own gravitational field, thus forming a global dynamical massive entity. In this paper, we show that in the case of a charged mass moving in an external electromagnetic field, the gravitational field of the global mass interacts with the electromagnetic potential through its current density. This interaction process must necessarily take a place in order for the global charge's lines of motion to satisfy a differential Finslerian system of equations whose form is similar to that of Riemannian geometry, as is the case for the neutral particle's geodesics. This result represents further evidence that the EGR model is an appropriate description of the mass and its subsequent gravitational field as a whole.

Contents:

Chapter 1 The Neutral Mass

§1.1 The EGR energy-momentum tensor 91

§1.2 The EGR geodesics 92

Chapter 2 The Charged Mass

§2.1 Charged density in an electromagnetic field 93

§2.2 The EGR electromagnetic current density. 94

Chapter 3 The EGR Differential Equations

§3.1 The EGR energy-momentum tensor of an electromagnetic field 96

§3.2 The EGR differential equations for the density flow lines of a charged mass 98

Conclusion 100

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Chapter 1. The Neutral Mass

§1.1. The EGR energy-momentum tensor. The EGR theory (Extended General Relativity) was introduced recently by the author [1] as a non-Riemannian extension of the General Theory of Relativity. Let us first recall the EGR source-free field equations

$$(G_{ab})_{\text{EGR}} = (R_{ab})_{\text{EGR}} - \frac{1}{2}\left[g_{ab}(R)_{\text{EGR}} - \frac{2}{3}\,J_{ab}\right] = \varkappa\,(T_{ab})_{\text{field}} \qquad (1.1)$$

with

$$J_{ab} = \partial_a J_b - \partial_b J_a\,, \qquad (1.2)$$

where $\varkappa$ is Einstein's constant, and $c = 1$. Here $(T_{ab})_{\text{field}}$ is the energy-momentum tensor of the EGR persistent field related to the tensor density $\sqrt{-g}\,(T^{ab})_{\text{field}} = (\mathcal{T}^{ab})_{\text{field}}$, which is determined through the equations

$$(\mathcal{T}^a_b)_{\text{field}} = \frac{1}{2\varkappa}\left[\partial_b\,\Gamma^e_{df}\,\frac{\partial\mathcal{H}}{\partial(\partial_a\Gamma^e_{df})} - \delta^a_b\,\mathcal{H}\right]. \qquad (1.3)$$

The invariant density $\mathcal{H}$ is given by $\mathcal{H} = \mathcal{R}^{ab}R_{ab}$ with the EGR Ricci tensor density $\mathcal{R}^{ab} = R^{ab}\sqrt{-g}$.

This background EGR field is assumed to be ever-present in vacuum. When matter is present, our previous studies [1–3] have led us to infer that the particle's (bare) mass density ρ is slightly modified, thus denoted hereinafter by ρ^*. The global quantity ρ^* is that part of the region surrounding the mass density, where Riemannian geometry increasingly dominates over the global one when asymptotically approaching the "bare" mass density ρ: it eventually becomes the single true geometry in the quasi "contact" situation. The reduction of the geometry in the immediate vicinity of the mass, can best be depicted by the transition of the surrounding EGR persistent field tensor density to the pseudo-tensor density ill-defined by Landau and Lifshitz, which conventionally describes the massive gravitational field

$$(\mathcal{T}^a_b)_{\text{LL}} = \frac{1}{2\varkappa}\left[\partial_b\,\mathcal{G}^{ec}\,\frac{\partial\mathcal{L}_{\text{E}}}{\partial(\partial_a\,\mathcal{G}^{ec})} - \delta^a_b\,\mathcal{L}_{\text{E}}\right], \quad \mathcal{G}^{ab} = \sqrt{-g}\,g^{ab}, \qquad (1.4)$$

where $\mathcal{L}_{\text{E}} = \sqrt{-g}\,g^{ab}\left(\left\{{e \atop ab}\right\}\left\{{d \atop de}\right\} + \left\{{d \atop ae}\right\}\left\{{e \atop bd}\right\}\right)$ is the Einstein Lagrangian density.

Therefore, the EGR theory enables to regard the quantity ρ^* as a generalized mass density including its own gravitational field, thus forming a single dynamical entity. Naturally, the correction brought to

Riemannian geometry is assumed to be weak. Hence, we write down the global energy-momentum tensor as

$$(T^{ab})_{\rm EGR} = \rho^{*}(u^{a}u^{b})_{\rm EGR} \tag{1.5}$$

or

$$(T^{ab})_{\rm EGR} = \rho\,(u^{a}u^{b})_{\rm EGR} + (t^{ab})_{\rm grav}\,, \tag{1.6}$$

where $(t^{ab})_{\rm grav}$ is the tensor of the gravitational field associated with the mass density, which classically corresponds (in Riemannian geometry) to the Landau-Lifshitz pseudo-tensor density (1.4).

The tensor $(t^{ab})_{\rm grav}$ is antisymmetric in accordance with the form of the EGR Einstein tensor $(G^{ab})_{\rm EGR}$ (1.1), and so is implicitly the tensor $(T^{ab})_{\rm EGR}$ (1.6).

The EGR Einstein tensor $(G_{ab})_{\rm EGR}$ obeys the conservation law

$$\left\{(R^{b}_{a})_{\rm EGR} - \frac{1}{2}\left[\delta^{b}_{a}(R)_{\rm EGR} - \frac{2}{3}\,J^{b}_{a}\right]\right\}_{{}',b} = 0\,. \tag{1.7}$$

Unlike in Riemannian geometry wherein covariant derivatives are constructed with the Christoffel symbols, the condition (1.7) utilizes the generally covariant derivatives $',$ (also denoted here by the symbol D) built from the global connection [1]

$$\Gamma^{d}_{ab} = \left\{{d \atop ab}\right\} + (\Gamma^{d}_{ab})_{J} = \left\{{d \atop ab}\right\} + \frac{1}{6}\left(\delta^{d}_{a}J_{b} + \delta^{d}_{b}J_{a} - 3\,g_{ab}\,J^{d}\right). \tag{1.8}$$

Therefore, in the absence of matter, the persistent field tensor we denote as $(T_{ab})_{\rm field}$ should be conserved according to (1.7)

$$\left[(T^{b}_{a})_{\rm field}\right]_{{}',b} = 0\,. \tag{1.9}$$

For the " massive" case, we have

$$\left[(T^{b}_{a})_{\rm EGR}\right]_{{}',b} = \left[\rho\,(u^{b}u_{a})_{\rm EGR} + (t^{b}_{a})_{\rm grav}\right]_{{}',b} = 0 \tag{1.10}$$

or, written equivalently,

$$\left[(T^{b}_{a})_{\rm EGR}\right]_{{}',b} = \left[\rho^{*}(u^{b}u_{a})_{\rm EGR}\right]_{{}',b} = 0\,. \tag{1.11}$$

§1.2. The EGR geodesics. A free neutral particle with mass m_0 classically follows a time-like geodesic according to the equation

$$\frac{d^{2}x^{b}}{ds^{2}} + \left\{{b \atop cd}\right\}\frac{dx^{c}}{ds}\frac{dx^{d}}{ds} = 0 \tag{1.12}$$

defined in a 4-Riemannian manifold equipped with a metric satisfying

$$g_{ab}\, u^a u^b = g^{ab}\, u_a u_b = 1\,, \tag{1.13}$$

where $u^a = \frac{dx^a}{ds}$ is the corresponding unit 4-vector (the world-velocity of the particle).

Following the EGR theory, inspection shows that the time-like geodesic equations shall have the same form

$$\left(\frac{d^2x^b}{ds^2}\right)_{\text{EGR}} + \Gamma^b_{cd}\left(\frac{dx^c}{ds}\frac{dx^d}{ds}\right)_{\text{EGR}} = 0\,. \tag{1.14}$$

Besides, the EGR world-velocity is slightly modified by the presence of the linear (non-square) form

$$dJ = f\,(J_b)\, dx^b, \tag{1.15}$$

so that the 4-velocity u^a becomes

$$(u^a)_{\text{EGR}} = \frac{dx^a}{\sqrt{ds^2 + dJ}}\,. \tag{1.16}$$

We also assume here that

$$g_{ab}\,(u^a u^b)_{\text{EGR}} = g^{ab}\,(u_a u_b)_{\text{EGR}} = 1\,. \tag{1.17}$$

Chapter 2. The Charged Mass

§2.1. Charged density in an electromagnetic field. With further contribution due to an external electromagnetic field, namely the Maxwell tensor F_{ab}, the geodesics of a particle with mass m_0 and charge e, are generated by the Finslerian curves which are known to be solutions of the Riemannian differential system

$$u^a\, \nabla_a u_b = \frac{e}{m_0}\, F_{ba}\, u^a = \frac{\mu}{\rho}\, F_{ba}\, u^a, \tag{2.1}$$

where ρ and μ are, respectively, the mass density and the charge density of the particle. An alternative form of (2.1) is given by the well-known formula

$$\frac{d^2x^b}{ds^2} + \left\{{b \atop cd}\right\} \frac{dx^c}{ds}\frac{dx^d}{ds} = \frac{\mu}{\rho}\, F^b_a\, \frac{dx^a}{ds}\,, \tag{2.2}$$

where the current vector is given by

$$j^a = \mu u^a. \tag{2.3}$$

The charged particle is said to satisfy a *Finslerian flow line.*

Classically, the general electromagnetic field energy-momentum tensor $(T_{ab})_{\text{elec}}$ is inferred from the Lagrangian

$$L = -\frac{1}{16\pi}\, F^{ab} F_{ab} - j_a A^a. \tag{2.4}$$

Henceforth, we use the Heaviside system of units where $\frac{1}{4}$ is substituted to the Gauss system $\frac{1}{16\pi}$.

As is well-known, the potential $A^a(x^a)$ is not a directly observable quantity, but is determined within a gradient

$$A'^a = A^a + \partial_a \psi\,. \tag{2.5}$$

Therefore it is customary to adopt a special gauge, which may be the *Lorentz gauge*. For the consistency of the theory, we keep this type of gauge throughout the text.

The tensor $(T_{ab})_{\text{elec}}$ is symmetrized, so as to yield

$$(T^{ab})_{\text{elec}} = \frac{1}{4}\, g^{ab} F_{cd} F^{cd} + F^{am} F_{m\cdot}^{\cdot\, b} + g^{ab} j_m A^m - j^a A^b. \tag{2.6}$$

However, the presence of sources violates (in general) the gauge invariance and also prevents this tensor from obeying the conservation law. This is why, in order to fit in the (symmetric) Einstein equations, one adds the symmetrized tensor (2.6) without source on the right-hand side of the Einstein-Maxwell field equations as

$$G_{ab} = \varkappa \left[\, \rho\, u_a u_b + (T_{ab})_{\text{elec}} \right]. \tag{2.7}$$

This somewhat arbitrary "adjustment" is true evidence of the rather awkward electromagnetic contribution to the classical field equations. In this sense, Riemannian geometry appears to be unable to thoroughly describe electrodynamics in the standard general relativistic theory. The problem can be cured by using the non-Riemannian connection as applied in the EGR theory, where the Einstein tensor is no longer symmetric. This intrinsic property allows for a straightforward and natural use of the canonical energy-momentum tensor of the electromagnetic field in the EGR Einstein field equations. As will be shown in §2.2 this canonical tensor is readily derived from a generalized Lagrangian density obtained in an analogous way as that used to deduce the EGR field equations.

§2.2. The EGR electromagnetic current density. Introducing the 4-potential A_a, the Maxwell tensor is written as

$$F_{ab} = \mathrm{D}_a A_b - \mathrm{D}_b A_a\,. \tag{2.8}$$

We proceed in strict analogy to the EGR stationary principle and set a Lagrangian density defined from the tensor and vector densities

$$\mathcal{F}^{ab} = \frac{\partial \mathcal{L}}{\partial F_{ab}}\,, \qquad \mathcal{I}^{a} = \frac{\partial \mathcal{L}}{\partial A_{a}}\,, \tag{2.9}$$

$$\mathcal{F}^{ab} = \sqrt{-g}\, F^{ab}, \tag{2.10}$$

$$\mathcal{I}^{a} = \sqrt{-g}\, I^{a}. \tag{2.11}$$

The varied action is then given by

$$\delta \mathcal{S} = \delta \int \mathcal{L}\, d^4x = 0\,. \tag{2.12}$$

For a variation δA_a, we further obtain

$$\delta \mathcal{S} = \int \left(\frac{1}{2} \frac{\partial \mathcal{L}}{\partial F_{ab}}\, \delta F_{ab} + \frac{\partial \mathcal{L}}{\partial A_a}\, \delta A_a \right) d^4x = 0\,, \tag{2.13}$$

while the variation of $\mathcal{L}$ is expressed by

$$\int \left(\frac{1}{2}\, \mathcal{F}^{ab} \delta F_{ab} + \mathcal{I}^{a} \delta A_a \right) d^4x = 0\,. \tag{2.14}$$

We integrate (2.14) by parts

$$\int \left[\frac{1}{2}\, \mathcal{F}^{ab} \left(\partial_a \delta A_b - \partial_b \delta A_a \right) + \mathcal{I}^{a} \delta A_a \right] d^4x =$$

$$= -\int \partial_b \left(\mathcal{F}^{ab} \delta A_a \right) d^4x + \int \left(\partial_b \mathcal{F}^{ab} + \mathcal{I}^{a} \right) \delta A_a\, d^4x = 0\,. \tag{2.15}$$

If the variations of A_a are zero on the integration boundary, the first integral yields no contribution. Then $\delta \mathcal{S} = 0$ implies

$$\partial_b \mathcal{F}^{ab} = -\,\mathcal{I}^{a}. \tag{2.16}$$

We clearly see that (2.8) and (2.16) represent the second group of Maxwell's equations given by the current density

$$\mathcal{I}^{a} = \sqrt{-g}\, I^{a}. \tag{2.17}$$

With a dynamical mass bearing its own gravitational field and having charge density μ, the global electromagnetic current density is obviously given by

$$I^{a} = \mu \left(u^{a} \right)_{\text{EGR}}\,, \tag{2.18}$$

which is collinear with the unit vector $(u^a)_{\text{EGR}}$.

Since the EGR description includes the gravitational field of the charged mass, it is natural to assume that this field interacts with the external electromagnetic field through the coupling between the current I_a and the potential A_a. This can be achieved by taking into account the term represented by the tensor

$$(T_{ab})_{\text{int}} = A_a I_b\,, \tag{2.19}$$

which is equivalent to saying that the dynamical mass ρ^* is affected by the interaction as follows

$$\rho^*(u_a u_b)_{\text{EGR}} \to \rho^*(u_a u_b)_{\text{EGR}} + (T_{ab})_{\text{int}} = (\rho^*)_{\text{int}}(u_a u_b)_{\text{EGR}}\,. \tag{2.20}$$

This assumption will find its full justification in §3.2.

Chapter 3. The EGR Differential Equations

§3.1. The EGR energy-momentum tensor of an electromagnetic field. Classically, the Lagrangian density displaying the current-potential coupling, and the Lagrangian itself are written as

$$\mathcal{L} = \frac{1}{2}\,\mathcal{F}^{ab}F_{ab} - A_a\mathcal{I}^a, \qquad L = \frac{1}{2}\,F^{ab}F_{ab} - A_a I^a. \tag{3.1}$$

Because we use the Heaviside system of units (see the note below formula 2.4 in page 94), the Lagrangian has the form

$$L = -\frac{1}{4}\,F^{ab}F_{ab} - A_a I^a. \tag{3.2}$$

The canonical energy-momentum tensor density $(\mathcal{E}^{ab})_{\text{EGR}}$ of the electromagnetic field is inferred from the Lagrangian density $\mathcal{L}$ (3.1). This (antisymmetric) tensor density has the usual generic form

$$(\mathcal{E}^{ab})_{\text{EGR}} = \left[\frac{\partial\mathcal{L}}{\partial(\partial_a A_m)}\right]\partial^b A_m - g^{ab}\mathcal{L}, \tag{3.3}$$

which has also a tensor counterpart such that

$$\sqrt{-g}\,\Theta^{ab} = (\mathcal{E}^{ab})_{\text{EGR}}\,. \tag{3.4}$$

Since the EGR Einstein tensor $(G_{ab})_{\text{EGR}}$ is not symmetric, it is thus most natural to apply the canonical energy-momentum tensor Θ_{ab} right away on the right-hand side of the EGR field equations. Therefore, in the case of a massive charged matter, the EGR field equations can be written with the electromagnetic field tensor as

$$(G_{ab})_{\mathrm{EGR}} = \varkappa \left[\rho^* (u_a u_b)_{\mathrm{EGR}} + \Theta_{ab} \right]. \tag{3.5}$$

The charged mass density is now represented by the global tensor

$$\begin{aligned}(T_{ab})_{\mathrm{EGR}} &= \rho\,(u_b u_a)_{\mathrm{EGR}} + (t_{ab})_{\mathrm{grav}} + \Theta_{ab} = \\ &= \rho^* (u_b u_a)_{\mathrm{EGR}} + \Theta_{ab}\,.\end{aligned} \tag{3.6}$$

Obviously, the persistent field tensor $(T_{ab})_{\mathrm{field}}$ does not appear explicitly on the right-hand side since we are here considering the "global massive" case. Also, the global mass ρ^* density is unaffected by the electromagnetic interaction (2.20) for the latter coupling is already included in the canonical tensor Θ_{ab}.

With the well-known classical identity

$$\frac{\partial\,(F^{kl}F_{kl})}{\partial(\partial_a A_m)} = 4F^{am}, \tag{3.7}$$

we obtain the canonical tensor Θ_{ab}, which is given in the EGR formulation by the formula

$$\Theta^{ab} = \frac{1}{4}\,g^{ab}F_{kl}F^{kl} - F^{am}\,\mathrm{D}^b A_m + g^{ab}I_m A^m \tag{3.8}$$

expressed with the EGR current density $I^m = \mu\,(u^m)_{\mathrm{EGR}}$ (2.18).

Using the tensor relations deduced from the equations of motion (2.16), and taking into account the antisymmetry of F_{ab}

$$\mathrm{D}_a F^{ba} = I^b, \tag{3.9}$$

we obtain a formula for the 4-divergence of Θ_{ab}. It is

$$\begin{aligned}\mathrm{D}_a\Theta^{ab} &= \frac{1}{2}\left(\mathrm{D}^b F_{kl}\right)F^{kl} - \left(\mathrm{D}_a F^{am}\right)\mathrm{D}^b A_m - F^{am}\mathrm{D}_b\mathrm{D}^b A_m + \\ &\quad + \left(\mathrm{D}^b I_m\right)A^m + I_m\mathrm{D}^b A^m = \\ &= -\frac{1}{2}\,\mathrm{D}^b\left(\mathrm{D}_k A_l + \mathrm{D}_l A_k\right)F^{kl} + \left(\mathrm{D}^b I_m\right)A^m,\end{aligned} \tag{3.10}$$

which, due to the antisymmetry of F_{kl}, obviously reduces to

$$\mathrm{D}_a\Theta^{ab} = (\mathrm{D}^b I_m)\,A^m. \tag{3.11}$$

We note in passing that the canonical tensor is conserved in the absence of electric current, which will be written as $(\Theta_{ab})_{\mathrm{free}}$.

In the latter case, the gauge change

$$A'_a \to A_a - \partial_a \psi \tag{3.12}$$

finally yields

$$(\Theta'^{ab})_{\text{free}} = (\Theta^{ab})_{\text{free}} - \mathrm{D}_k \left(F^{ak}\, \partial^b \psi\right) \tag{3.13}$$

having $\mathrm{D}_k F^{ka} = I^a = 0$ taken into account.

Hence, $(\Theta^{ab})_{\text{free}}$ is not gauge invariant, but the second divergence term yields no contribution upon integration, and thus $(\Theta^{ab})_{\text{free}}$ is here a conserved quantity. Therefore, this (antisymmetric) canonical source-free tensor should be the appropriate candidate for the EGR field equations (3.5), provided we use the modified global mass density $(\rho^*)_{\text{int}}$.

In place of (3.6), we eventually write the equivalent formula

$$(T_{ab})_{\text{EGR}} = (\rho^*)_{\text{int}} (u_a u_b)_{\text{EGR}} + (\Theta_{ab})_{\text{free}}\,. \tag{3.14}$$

Unlike in Riemannian geometry, we clearly see that the EGR formulation allows us to include the electromagnetic source contribution represented by $(\rho^*)_{\text{int}}$, in the EGR Einstein-Maxwell equations.

In the absence of matter, the EGR energy-momentum tensor of the pure electromagnetic field is simply

$$(T_{ab})_{\text{EGR}} = (T_{ab})_{\text{field}} + (\Theta_{ab})_{\text{free}}\,. \tag{3.15}$$

§3.2. The EGR differential equations for the density flow lines of a charged mass. Our final aim is to find a differential system satisfied by the global charge, whose form is similar to the Riemannian system (2.2), as is the case for a neutral mass. To this effect, we first revert to the global energy-momentum as written in (3.6), for which the conservation law is given by

$$\left[\,\rho^*(u^b u_a)_{\text{EGR}} + \Theta^b_a\,\right]_{';b} = 0\,. \tag{3.16}$$

We introduce the vector K_b defined by

$$\rho^* K_b = \mathrm{D}_a \Theta^a_b = (\mathrm{D}_b I_m)\, A^m. \tag{3.17}$$

For that, we write the right-hand side as follows

$$\mathrm{D}_b (I_m A^m) - I_m \mathrm{D}_b A^m = (\mathrm{D}_b I_m)\, A^m, \tag{3.18}$$

and noting that

$$I_m \mathrm{D}_b A^m = \frac{1}{2}\, I_m (\mathrm{D}_b A^m - \mathrm{D}^m A_b) \tag{3.19}$$

the conservation conditions for the global tensor take the form

$$\mathrm{D}_a \left[\, \rho^* (u^a u_b)_{\mathrm{EGR}} \right] = -\, \rho^* K_b = $$
$$= \mathrm{D}_b \left(I^m A_m \right) - \frac{1}{2}\, I_m \left(\mathrm{D}_b A^m - \mathrm{D}^m A_b \right). \quad (3.20)$$

Taking into account the formula

$$(u^b)_{\mathrm{EGR}}\, \mathrm{D}_a (u_b)_{\mathrm{EGR}} = 0\,, \quad (3.21)$$

which follows from differentiating (1.17), we find, after multiplying through (3.20) by $(u^b)_{\mathrm{EGR}}$,

$$\mathrm{D}_a \left[\, \rho^* (u^a)_{\mathrm{EGR}} \right] = -\, \rho^* K_a (u^a)_{\mathrm{EGR}}\,. \quad (3.22)$$

The continuity equation is thus expressed by

$$\mathrm{D}_a \left[\, \rho^* (u^a)_{\mathrm{EGR}} \right] = -\, \mu\, (u^a)_{\mathrm{EGR}} \times$$
$$\times \left\{ \mathrm{D}_a \left[(u_m)_{\mathrm{EGR}} A^m \right] - \frac{1}{2}\, (u_m)_{\mathrm{EGR}} \big(\mathrm{D}_a A^m - \mathrm{D}^m A_a \big) \right\}. \quad (3.23)$$

After some simplifications, we arrive at the differential system determining the flow lines of the charged particle

$$(u^a)_{\mathrm{EGR}}\, \mathrm{D}_a (u_b)_{\mathrm{EGR}} = \left[\, \delta^a_b - (u^a u_b)_{\mathrm{EGR}} \right] \times$$
$$\times\, \frac{\mu}{\rho^*} \left\{ -\mathrm{D}_a \left[(u^m)_{\mathrm{EGR}} A_m \right] + \frac{1}{2}\, F_{am} (u^m)_{\mathrm{EGR}} \right\}. \quad (3.24)$$

Now, if we assume that the dynamical mass density ρ^* interacting with the potential A_m is modified so that

$$-\, K_b (\rho^*)_{\mathrm{int}} = \mathrm{D}_a \left[(\rho^*)_{\mathrm{int}} (u^a u_b)_{\mathrm{EGR}} \right] = -\mu\, \mathrm{D}_b \left[(u_m)_{\mathrm{EGR}} A^m \right], \quad (3.25)$$

we eventually obtain

$$(u^a)_{\mathrm{EGR}}\, \mathrm{D}_a (u_b)_{\mathrm{EGR}} = \left[\, \delta^a_b - (u^a u_b)_{\mathrm{EGR}} \right] \frac{1}{2}\, F_{am} (u^m)_{\mathrm{EGR}}\,. \quad (3.26)$$

These equations are to be compared to the Riemannian differential system

$$u^a\, \nabla_a u_b = \big(\delta^a_b - u^a u_b \big)\, \frac{\mu}{\rho}\, F_{am} u^m, \quad (3.27)$$

which reduces to the well-known classical equations $u^a \nabla_a u_b = \frac{\mu}{\rho}\, F_{ba} u^a$ (2.1) since F^{am} is antisymmetric.

In other words, imposing the above-mentioned conditions, the Finslerian trajectory of the global charged density will now satisfy the differential system

$$\left(\frac{d^2x^b}{ds^2}\right)_{\rm EGR} + \Gamma^b_{cd}\left(\frac{dx^c}{ds}\frac{dx^d}{ds}\right)_{\rm EGR} = \frac{1}{2}\,\frac{\mu}{(\rho^*)_{\rm int}}\,F^b_a\left(\frac{dx^a}{ds}\right)_{\rm EGR}. \qquad (3.28)$$

Within the numerical factor $\frac{1}{2}$, this EGR formulation is formally similar to the differential system of Riemannian geometry satisfied by the charged mass density trajectory according to the classical General Relativity.

Conclusion. Upon imposing the Lorentz gauge, we are able to generalize some basic principles of electrodynamics via the EGR theory. In the EGR formulation, three main results readily emerge:

1) Unlike in Riemannian geometry, the (antisymmetric) canonical electromagnetic energy-momentum tensor (3.8), as inferred from the Lagrangian (3.2), can be readily used in the EGR field equations, without post-symmetrization adjustment;
2) The dynamical global charged mass (current) interacting with the electromagnetic field implicitly appears in the EGR field equations. This result is impossible to express in Riemannian geometry (classical General Relativity), which stands so far as a profound loss of generality in the metric theory;
3) With the 2nd condition outlined above in this list, we are eventually able to infer the differential system (3.28) obeyed by the global charged mass, which is formally similar to the differential system (2.2) introduced according to Riemannian geometry, a similarity already existent between the Riemannian and EGR geodesics for the neutral mass, as given by (1.12) and (1.14). This last result gives us further evidence to substantiate the EGR model as representing the motion of a mass dynamically bearing its own gravitational field.

In conclusion, therefore, a last important point should be outlined here. Either the geodesic equations (1.14) for a neutral particle, or the Finslerian equations for a charged particle (3.28) (each system with its own corresponding gravitational field), does not distinguish antimatter from matter. The EGR model can, however, be adequately used to interpret the fermionic-antifermionic symmetry as postulated by Louis de Broglie, and generalized in [4].

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A Hydrodynamical Geometrization of Matter and Chronometricity in General Relativity

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Abstract: In this work, we outline a new complementary model of the relativistic theory of an inhomogeneous, anisotropic universe which was first very extensively proposed by Abraham Zelmanov to encompass all possible scenarios of cosmic evolution within the framework of the classical General Relativity, especially through the development of the mathematical theory of chronometric invariants. In doing so, we propose a fundamental model of matter as an intrinsic flexural geometric segment of the cosmos itself, i.e., matter is modelled as an Eulerian hypersurface of world-points that moves, deforms, and spins along with the entire Universe. The discrete nature of matter is readily encompassed by its representation as a kind of discontinuity curvature with respect to the background space-time. In addition, our present theoretical framework provides a very natural scheme for the unification of physical fields. An immediate scale-independent particularization of our preliminary depiction of the physical plenum is also considered in the form of an absolute monad model corresponding to a universe possessing absolute angular momentum.

Contents:

§1. Introduction 102
§2. The proposed geometrization of matter: a cosmic monad 105
§3. Reduction to the pure monad model 111
§4. Hydrodynamical unification of physical fields 114
§5. A Machian monad model of the Universe 116
§6. Conclusion 119

Dedicated to Abraham Zelmanov

§1. Introduction. The relativistic theory of a fully inhomogeneous, anisotropic universe in the classical framework of Einstein's General Theory of Relativity has been developed to a fairly unprecedented, over-arching extent by the general relativist and cosmologist Abraham Zelmanov [1]. The ingenious methodology of Zelmanov has been profoundly utilized and developed in several interesting physical situations, shedding further light on the intrinsic and extensive nature of the clas-

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sical General Relativity as a whole (see [2]).

By the phrase "classical General Relativity", we wish to emphasize that only space-time and gravitational fields have been genuinely, cohesively geometrized by the traditional Einsteinian theory. Nevertheless, the construction of the mathematical theory of chronometric invariants by Zelmanov enables one to treat General Relativity pretty much in the context of some kind of four-dimensional continuum mechanics of the very substance (plenum) of space-time geometry itself.

We factually note, in passing, that several independent theoretical approaches to the geometric unification of space-time, matter, and physical fields, in both the extensively classical sense and the non-classical sense, have been constructed by the Author elsewhere (for instance, see [3] and the bibliographical list of the Author's preceding works — diverse as they are — therein).

In the present work, we are singularly concerned with the methodology originally outlined by Zelmanov. Nevertheless, fully acquainted with the powerful depth, elegance, and beauty of his work, we shall still present some newly emerging ideas by first-principle construction, as well as some well-established understandings afresh, while uniquely situating ourselves in the alleyway wherefrom both the cosmos and the classical General Relativity are insightfully envisioned by Zelmanov.

As such, we shall theoretically fill a few gaps in the fabric of the classical General Relativity in general, and of Zelmanov's methodology in particular, by proposing a fundamental hydrodynamical model of matter, so as to possibly substantiate the material structure of the observer in common with the preferred, stable cosmic reference frame with respect to which the observer is at rest, i.e., one that co-moves, co-deforms, and co-rotates with respect to the entire Universe.

Indeed, we shall proceed first by geometrizing matter and discovering a natural way to reflectively superimpose the small-scale picture upon the entire Universe, yielding a unified description of the observer and the cosmos.

In the very general sense (far from the usual homogeneous, isotropic cosmological situations), we may note at this point that not all observers can automatically be qualified as fundamental observers, i.e. "observational monads", with respect to whose observation the structure of the Universe intrinsically appears the way it is observed by them.

Such, of course, is true also for observers assuming a homogeneous, isotropic universe and observing it accordingly. However, in certain cases incorporating, e.g., the absolute rotation of the universe, the problem of true interiority (and structural totality) arises in the sense that we can no longer recognize certain innate properties of the universe in the reference frames specific to homogeneous, isotropic models only. Such frames may be slowly translating and deforming to keep themselves at the natural expansive rate of the universe, but in the presence of self-rotation (intrinsic angular momentum), a physical system is quite something else to be accounted for in itself. For, if certain elementary particles (as we know them) are truly elementary, we shall know the total sense of observation of the Universe also from within their (common) interior and ultimately discover that, irrespective of scales, the Universe is self-contained in their very existence.

Now, recalling that which lies at the heart of the theory of chronometric invariants, we may posit further that the interior (and the total possible exterior) of the Universe can only be known by a rather advanced non-holonomic observer, i.e., one who is not merely "incidental" to the mesoscopic scale of (seemingly homogeneous) ordinary things, but one who builds his system of reference with respect to the interior and exterior of things in the required extreme limits, i.e., by rather direct in-depth cognition of the logically self-possible meta-Universe, beyond any self-limited experimental set-up. In other words, the totality of the laws of cognition is intrinsic to such an observer endowed with a "syntactical totality of logical operators" (a whole contingency of self-reflexive logical grammar). This, in turn, necessarily belongs to the interior of the directly observable (perceptual) Universe. One can then see how this substantially differs from a mere "bootstrap" universe.

Hence, regarding observation, our "anthropic principle with further self-qualification" is true only for observers dynamically situating themselves in certain unique non-holonomic frames of reference bearing the specific characteristics of motion of very elementary microscopic objects (such as certain elementary particles) and macroscopic objects exhibiting natural chronometricity with respect to the whole Universe (such as certain spinning stars, planets, galaxies, and metagalaxies). This, then, would be true for individual observers as well as an aggregate of common observers — such as those situated on a special rotating (planetary) islet of mass — in their own unique ("universally preferred") non-holonomic coordinate systems.

Such observers are truly situated at the world-points of the respective Eulerian hypersurfaces (representing matter) in common with the entire non-holonomic, inhomogeneous, anisotropic Universe. This is because, while "inhering" in matter itself, they automatically possess all the geometric material configurations intrinsic to both matter itself and the entire Universe.

In our theory, as we shall see, the projective chronometric structure plays the role of the geometrized non-Abelian gauge field strength. Hence, any natural extension of the study will center around the corresponding emphasis that the inner constitution of elementary particles cannot be divorced from chronometricity (the way it is hydrodynamically geometrized here). This, like in the original case of Zelmanov, gives one the penetrating confidence to speak of elementary particles, in addition to large-scale objects, purely in the framework of the chronometric General Relativity, as if without having to mind the disparities involving scales (of particles and galaxies).

Here, chronometricity is geometrized in such a way that co-substantial motion results, both hydrodynamically and geometrically, from the fundamental properties of the extrinsic curvature of the material hypersurface (i.e., matter itself).

In addition, the Yang-Mills curvature is generalized by the presence of the asymmetric extrinsic curvature, as in [5]. Only in pervasive flatness does it go into the usual Yang-Mills form of the Standard Model (whose background space-time is Minkowskian). We shall not employ the full form of the particular Finslerian connection as introduced in [3], but only the respective metric-compatible part, with the corresponding geodesic equation of motion intrinsically generating the generally covariant Lorentz equation.

Hence, while encompassing the elasticity of space-time, we shall further advance the notion of a discontinuous Eulerian hypersurface such that it geometrically represents matter and chronometricity at once, and such that it may be applied to any cosmological situation independently of scales.

Indeed, as we shall see, the Machian construction (see §5 herein) is a special condition for "emergent inertia", without having to invoke both Newtonian absolute (external) empty space and a distant reference frame. Rather, the whole process is meant to be topologically scale-independent. An alternative objective of the present approach, therefore, is such that the structure of General Relativity, when developed (generalized) this way, can apparently meet that of quantum theory in a parallel fashion.

§2. The proposed geometrization of matter: a cosmic monad. Let us consider an arbitrary orientation of a mobile, spinning hypersurface $\mathrm{C}^3(t) = \partial\Sigma^4$ as the boundary of the world-tube Σ^4 of geodesics in the background Universe M^4. Denoting the regular boundary by $\mathrm{B}^3(t)$ and the discontinuity hypersurface cutting through Σ^4 by Υ, we see that $C^3(t) = B^3(t) \cup \Upsilon$. We emphasize that $C^3(t)$ is a natural geometric segment of M^4, i.e., it is created purely by the dynamics of the intrinsic (and global) curvature and torsion of M^4. This is to the extent that the unit normal vector with respect to $C^3(t)$ is immediately given by the world-velocity $u^\alpha(s)$ along the world-line s.

We call $C^3(t)$ a monad, i.e., a substantive Eulerian structure of matter. As we shall see, this dynamical monad model is fully intrinsic to the fabric of space-time, i.e., inseparable from (not external to) the intrinsic structure of the Universe, thereby allowing us to incorporate the subsequent geometrization of matter (and material fields) into Einstein's field equation.

Our substantial depiction of matter filling the cosmos also implies the wave-like nature of the hypersurface $C^3(t)$, for the velocity field of the points of $C^3(t)$ — representing individual group particles — is no longer singly oriented. This allows us to project the fundamental material structure pervasively outward — onto the Universe itself. Consequently, this model readily applies to all sorts of observers, other than just a comoving one (whose likeness we shall especially refer to as the "purely monad observer").

As we know, the infinitesimal world-line, along which $C^3(t)$ moves, is explicitly given by the metric tensor $g_{\alpha\beta}(x)$ of M^4 as

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{00}\, dx^0 dx^0 + 2\, g_{0i}\, dx^0 dx^i + g_{ik}\, dx^i dx^k = \\ = c^2 d\tau^2 - d\sigma^2, \tag{2.1}$$

where we denote the speed of light as c. The proper time, the generally non-holonomic, evolutive spatial segment (the hypersurface segment), the metric tensor of the hypersurface, and the linear velocity of space rotation (i.e., of material spin) are respectively given by*

$$d\tau = \frac{g_{0\alpha}\, dx^\alpha}{c\sqrt{g_{00}}} = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i = \sqrt{g_{00}}\, dt - \frac{1}{c^2}\, v_i\, dx^i, \tag{2.2}$$

$$d\sigma = \sqrt{h_{ik}\, dx^i dx^k}\,, \tag{2.3}$$

$$h_{ik} = -\, g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}} = -\, g_{ik} + \frac{1}{c^2}\, v_i v_k\,, \tag{2.4}$$

$$v_i = -\, c\, \frac{g_{0i}}{\sqrt{g_{00}}}\,. \tag{2.5}$$

*Einstein's summation convention is utilized with space-time Greek indices running from 0 to 3 and projective material-spatial Latin indices from 1 to 3.

Denoting the unit normal vector of the material hypersurface by N^α, we see that $N^\alpha = u^\alpha = \frac{dx^\alpha}{ds}$, and especially that

$$N_i = u_i = -\frac{1}{c}\,v_i\,. \tag{2.6}$$

In a simplified matrix representation, we therefore have

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & N_i\sqrt{g_{00}} \\ N_i\sqrt{g_{00}} & -h_{ik} + N_i N_k \end{pmatrix}. \tag{2.7}$$

Now, the fundamental projective relation between the background space-time metric $g_{\alpha\beta}(x)$ and the global material metric $h_{ik}(x,u)$ is readily given as

$$g_{\alpha\beta} = -h_{\alpha\beta} + u_\alpha u_\beta\,, \tag{2.8}$$

where, with $f(v^i, dt) \rightarrow v^i f(dt)$ and $f\big(v_i, \frac{\partial}{\partial t}\big) \rightarrow v_i f\big(\frac{\partial}{\partial t}\big)$,

$$h_{\alpha\beta} = \frac{\partial Y^i}{\partial x^\alpha}\frac{\partial Y^k}{\partial x^\beta}\,(-g_{ik})\,, \tag{2.9}$$

$$dY^i = dx^i + f(v^i, dt)\,, \tag{2.10}$$

$$\frac{\partial}{\partial Y^i} = \frac{\partial x^\alpha}{\partial Y^i}\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^i} + f\left(v_i, \frac{\partial}{\partial t}\right), \tag{2.11}$$

$$h^\alpha_\beta = -\delta^\alpha_\beta + u^\alpha u_\beta\,, \qquad h^\alpha_\gamma h^\gamma_\beta = \delta^\alpha_\beta - u^\alpha u_\beta\,, \tag{2.12}$$

$$h^i_\alpha = h^\beta_\alpha\frac{\partial Y^i}{\partial x^\beta} = -\frac{\partial Y^i}{\partial x^\alpha}\,, \tag{2.13}$$

$$h^\alpha_i h^i_\beta = \delta^\alpha_\beta - u^\alpha u_\beta\,, \qquad h^i_\alpha h^\alpha_k = \delta^i_k\,, \tag{2.14}$$

$$h_{\alpha\beta}\,u^\beta = 0\,, \qquad h^i_\alpha\,u^\alpha = 0\,. \tag{2.15}$$

Let us represent the natural basis vector of M^4 by $\bar{g}_\alpha$ and that of $\mathrm{C}^3(t)$ by $\bar{\omega}_i$. We immediately obtain the generally asymmetric extrinsic curvature of $\mathrm{C}^3(t)$ through the inner product

$$Z_{ik} = \left\langle u, \frac{\partial\bar{\omega}_i}{\partial Y^k}\right\rangle \tag{2.16}$$

i.e.,

$$Z_{ik} = -u_\alpha\nabla_k h^\alpha_i = -h^\alpha_i h^\beta_k\,\nabla_\beta u_\alpha\,, \tag{2.17}$$

where ∇ denotes covariant differentiation, i.e., for an arbitrary tensor field $Q^{ab\dots}_{cd\dots}(x)$ and metric-compatible connection form $\Gamma^a_{mk}(x)$, presented herein with arbitrary indexing,

$$\nabla_k Q^{ab\ldots}_{cd\ldots} = \frac{\partial Q^{ab\ldots}_{cd\ldots}}{\partial x^k} + \Gamma^a_{mk} Q^{mb\ldots}_{cd\ldots} + \Gamma^b_{mk} Q^{am\ldots}_{cd\ldots} + \ldots - \Gamma^m_{ck} Q^{ab\ldots}_{md\ldots} - \Gamma^m_{dk} Q^{ab\ldots}_{cm\ldots} - \ldots\,, \tag{2.18}$$

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{am} \left(\frac{\partial g_{mb}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^m} + \frac{\partial g_{cm}}{\partial x^b} \right) + \Gamma^a_{[bc]} - g^{am} \left(g_{bn} \Gamma^n_{[mc]} + g_{cn} \Gamma^n_{[mb]} \right), \tag{2.19}$$

$$\frac{\mathrm{D} Q^{ab\ldots}_{cd\ldots}}{ds} = u^e\, \nabla_e Q^{ab\ldots}_{cd\ldots}\,. \tag{2.20}$$

Henceforth, round and square brackets on indices shall indicate symmetrization and anti-symmetrization, respectively.

Hence, we see that the extrinsic curvature tensor of the material hypersurface is uniquely expressed in terms of the four-dimensional velocity gradient tensor given by the expression

$$\varphi_{\alpha\beta} = \nabla_\beta\, u_\alpha\,, \tag{2.21}$$

i.e.,

$$Z_{ik} = -\, h^\alpha_i\, h^\beta_k\, \varphi_{\alpha\beta}\,. \tag{2.22}$$

This way, we have indeed geometrized the tensor of the rate of material deformation $\Theta_{\alpha\beta}$ and the tensor of material vorticity $\omega_{\alpha\beta}$, as can be seen from the respective symmetric and anti-symmetric expressions below:

$$Z_{(ik)} = -\, h^\alpha_i\, h^\beta_k\, \Theta_{\alpha\beta}\,, \qquad Z_{[ik]} = -\, h^\alpha_i\, h^\beta_k\, \omega_{\alpha\beta}\,, \tag{2.23}$$

where

$$\Theta_{\alpha\beta} = \frac{1}{2} \left(\nabla_\beta\, u_\alpha + \nabla_\alpha\, u_\beta \right), \qquad \omega_{\alpha\beta} = \frac{1}{2} \left(\nabla_\beta\, u_\alpha - \nabla_\alpha\, u_\beta \right). \tag{2.24}$$

Meanwhile, noting immediately that

$$\nabla_k\, h^\alpha_i = -\, Z_{ik}\, u^\alpha, \tag{2.25}$$

we obtain the following relation:

$$\frac{\partial h^\alpha_i}{\partial Y^k} = \Omega^p_{ik}\, h^\alpha_p - \Gamma^\alpha_{\beta\gamma}\, h^\beta_i\, h^\gamma_k - Z_{ik}\, u^\alpha. \tag{2.26}$$

Both $\Omega^p_{ik}(Y^p)$ of $\mathrm{C}^3(t)$ and $\Gamma^\alpha_{\beta\gamma}(x)$ of M^4 are generally asymmetric, non-holonomic connection forms. We see that they are related to each other through the following fundamental relations:

$$\Omega^{p}_{ik} = h^{p}_{\alpha}\,\frac{\partial h^{\alpha}_{i}}{\partial Y^{k}} - h^{p}_{\alpha}\,\Gamma^{\alpha}_{\beta\gamma}\,h^{\beta}_{i}\,h^{\gamma}_{k}\,, \tag{2.27}$$

$$\Gamma^{\alpha}_{\beta\gamma} = h^{\alpha}_{i}\,\frac{\partial h^{i}_{\beta}}{\partial x^{\gamma}} - h^{\alpha}_{p}\,\Omega^{p}_{ik}h^{i}_{\beta}h^{k}_{\gamma} + Z_{ik}\,h^{i}_{\beta}h^{k}_{\gamma}u^{\alpha} + u^{\alpha}\,\frac{\partial u_{\beta}}{\partial x^{\gamma}} - Z^{i\cdot}_{\cdot k}\,h^{\alpha}_{i}h^{k}_{\gamma}u_{\beta}\,. \tag{2.28}$$

The associated curvature tensor of $\mathrm{C}^3(t)$, $R^{i\cdot\cdot\cdot}_{\cdot jkl}(\Omega^{i}_{jl})$, and that of M^4, $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\rho\gamma}(\Gamma^{\alpha}_{\beta\gamma})$, are then respectively given by

$$R^{i\cdot\cdot\cdot}_{\cdot jkl} = \frac{\partial\Omega^{i}_{jl}}{\partial Y^{k}} - \frac{\partial\Omega^{i}_{jk}}{\partial Y^{l}} + \Omega^{p}_{jl}\,\Omega^{i}_{pk} - \Omega^{p}_{jk}\,\Omega^{i}_{pl}\,, \tag{2.29}$$

$$R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\rho\gamma} = \frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\alpha}_{\beta\rho}}{\partial x^{\gamma}} + \Gamma^{\tau}_{\beta\gamma}\Gamma^{\alpha}_{\tau\rho} - \Gamma^{\tau}_{\beta\rho}\Gamma^{\alpha}_{\tau\gamma}\,, \tag{2.30}$$

where, as usual,

$$\begin{aligned}\big(\nabla_l\nabla_k - \nabla_k\nabla_l\big)Q^{ab\ldots}_{cd\ldots} &= \\ &= R^{m\cdot\cdot\cdot}_{\cdot ckl}Q^{ab\ldots}_{md\ldots} + R^{m\cdot\cdot\cdot}_{\cdot dkl}Q^{ab\ldots}_{cm\ldots} + \cdots - R^{a\,\cdot\cdot\cdot}_{\cdot mkl}Q^{mb\ldots}_{cd\ldots} - \\ &\quad - R^{b\,\cdot\cdot\cdot}_{\cdot mkl}Q^{am\ldots}_{cd\ldots} - \cdots - 2\Gamma^{m}_{[kl]}\nabla_m\,Q^{ab\ldots}_{cd\ldots}\,,\end{aligned} \tag{2.31}$$

$$\big(\nabla_b\nabla_a - \nabla_a\nabla_b\big)\,\phi = -2\Gamma^{c}_{[ab]}\,\nabla_c\,\phi\,, \tag{2.32}$$

where ϕ is an arbitrary scalar field.

At this point, we obtain the complete projective relations between the background space-time geometry and the geometric material space. The relations are as follows:

$$R_{ijkl} = Z_{ik}\,Z_{jl} - Z_{il}\,Z_{jk} + h^{\alpha}_{i}\,h^{\beta}_{j}\,h^{\rho}_{k}\,h^{\gamma}_{l}\,R_{\alpha\beta\rho\gamma} + S_{\alpha jkl}\,h^{\alpha}_{i}\,, \tag{2.33}$$

$$\nabla_l\,Z_{ik} - \nabla_k\,Z_{il} = u^{\alpha}\,h^{\beta}_{i}h^{\rho}_{k}h^{\gamma}_{l}\,R_{\alpha\beta\rho\gamma} - 2\,\Omega^{p}_{[kl]}\,Z_{ip} + u^{\alpha}\,S_{\alpha ikl}\,. \tag{2.34}$$

In terms of the curvature tensor $R^{i\cdot\cdot\cdot}_{\cdot jkl}(\Omega^{i}_{jl})$ of $\mathrm{C}^3(t)$, and that of M^4, which is $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\rho\gamma}(\Gamma^{\alpha}_{\beta\gamma})$, with the segmental torsional curvature (incorporating possible analytical discontinuities as well) given by

$$S^{\alpha\cdot\cdot\cdot}_{\cdot ijk} = \frac{\partial}{\partial Y^{j}}\left(\frac{\partial h^{\alpha}_{i}}{\partial Y^{k}}\right) - \frac{\partial}{\partial Y^{k}}\left(\frac{\partial h^{\alpha}_{i}}{\partial Y^{j}}\right) + \Gamma^{\alpha}_{\beta\gamma}\,h^{\beta}_{i}\left(\frac{\partial h^{\gamma}_{j}}{\partial Y^{k}} - \frac{\partial h^{\gamma}_{k}}{\partial Y^{j}}\right). \tag{2.35}$$

Now, we can four-dimensionally express the (generalized generally covariant) gravitational force F_{α}, the spatial deformation $D_{\alpha\beta}$, and the angular momentum $A_{\alpha\beta}$ in terms of our geometrized material deformation and material vorticity as follows:

$$F_\alpha = 2c^2 u^\beta \omega_{\alpha\beta}\,, \tag{2.36}$$

$$D_{\alpha\beta} = c h^\mu_\alpha h^\nu_\beta\, \Theta_{\mu\nu}\,, \tag{2.37}$$

$$A_{\alpha\beta} = c h^\mu_\alpha h^\nu_\beta\, \omega_{\mu\nu}\,, \tag{2.38}$$

such that

$$\Phi_{\alpha\beta} = c h^\mu_\alpha h^\nu_\beta\, \varphi_{\mu\nu} = D_{\alpha\beta} + A_{\alpha\beta}\,, \tag{2.39}$$

$$D_{ik} = h^\mu_i h^\nu_k\, \Phi_{(\mu\nu)}\,, \tag{2.40}$$

$$A_{ik} = h^\mu_i h^\nu_k\, \Phi_{[\mu\nu]}\,. \tag{2.41}$$

As a result, we obtain the geometrized dynamical relation

$$R_{ijkl} = h^\alpha_i h^\beta_j h^\rho_k h^\gamma_l \left(R_{\alpha\beta\rho\gamma} + \varphi_{\alpha\rho}\varphi_{\beta\gamma} - \varphi_{\alpha\gamma}\varphi_{\beta\rho}\right) + h^\alpha_i S_{\alpha jkl}\,. \tag{2.42}$$

Furthermore, let us introduce the Eulerian (substantive) curvature of the material hypersurface which satisfies all the natural symmetries of the curvature and torsion tensors of the background space-time as follows:

$$F_{ijkl} = h^\alpha_i h^\beta_j h^\rho_k h^\gamma_l R_{\alpha\beta\rho\gamma} + h^\alpha_i S_{\alpha jkl}\,. \tag{2.43}$$

We immediately see that

$$\begin{aligned} F_{ijkl} = R_{ijkl} - \frac{1}{c}\big(&D_{ik}D_{jl} - D_{il}D_{jk} + A_{ik}A_{jl} - A_{il}A_{jk} + \\ &+ D_{ik}A_{jl} - D_{il}A_{jk} + A_{ik}D_{jl} - A_{il}D_{jk}\big)\,, \end{aligned} \tag{2.44}$$

and, in addition, we also obtain the inverse projective relations

$$\begin{aligned} &R_{\mu\nu\rho\sigma} - R_{\lambda\nu\rho\sigma}u^\lambda u_\mu - R_{\mu\lambda\rho\sigma}u^\lambda u_\nu - R_{\mu\nu\lambda\sigma}u^\lambda u_\rho - \\ &\quad - R_{\mu\nu\rho\lambda}u^\lambda u_\sigma - R_{\lambda\nu\kappa\sigma}u^\lambda u^\kappa u_\mu u_\rho - R_{\lambda\nu\rho\kappa}u^\lambda u^\kappa u_\mu u_\sigma - \\ &\quad - R_{\mu\lambda\kappa\sigma}u^\lambda u^\kappa u_\nu u_\rho - R_{\mu\lambda\rho\kappa}u^\lambda u^\kappa u_\nu u_\sigma = \\ &\quad = h^j_\nu h^k_\rho h^l_\sigma \big(h^i_\mu (R_{ijkl} - Z_{ik}Z_{jl} + Z_{il}Z_{jk}) + S_{\mu jkl} - u_\mu u^\lambda S_{\lambda jkl}\big)\,, \end{aligned} \tag{2.45}$$

$$\begin{aligned} &R_{\lambda\mu\nu\rho}u^\lambda - R_{\lambda\mu\kappa\rho}u^\lambda u^\kappa u_\nu - R_{\lambda\mu\nu\kappa}u^\lambda u^\kappa u_\rho = \\ &\quad = -h^i_\mu h^k_\nu h^l_\rho \left(\nabla_l Z_{ik} - \nabla_k Z_{il} + 2\Omega^p_{[kl]} Z_{ip} - S_{\lambda ikl}u^\lambda\right). \end{aligned} \tag{2.46}$$

The complete geometrization of matter in this hydrodynamical approach represents a continuum mechanical description of space-time where the extrinsic curvature of any material hypersurface manifests itself as the gradient of its velocity field. As such, the geometric field equations simply consist in specifying the world-velocity of the moving matter (especially directly from reading off the components of the fundamental metric tensor). The acquisition of individual particles, as a special case of the more general group particles, is immediately at hand when the material hypersurface enclosing a volumetric segment of the cosmos is small enough, i.e., in this case the particles are ordinary infinitesimal space-time points translating and spinning in common with the deforming and spinning Universe on the largest scale.

§3. Reduction to the pure monad model. Having formulated the general structure of our scheme for the substantive geometrization of matter (as well as physical fields, essentially by way of our preceding works as listed in [3]) in the preceding section, we can now explicitly arrive at the cosmological picture of Zelmanov for general relativistic dynamics, i.e., the theory of chronometric invariants.

Much in parallel with Yershov [4], we may simply state the strong monad model of the cosmos of Zelmanov as follows:

1) The Universe as a whole spins, inducing the spin of every elementary constituent in it;
2) The Universe is intrinsically inhomogeneous, anisotropic, and nonholonomic, giving rise to its diverse elementary constituents (i.e., particles) on the microscopic scale, including its specific fundamental properties (e.g., mass, charge, and spin);
3) The small-scale structure of the Universe is simply holographic ("isomorphic") to the large-scale cosmological structure, thereby rendering the Universe truly self-contained;
4) The linear velocity (or momentum) of any microscopic or macroscopic object is essentially induced by the global spin of the Universe, such that the individual motion of matter is none other than the segmental motion of the Universe.

We shall refer to the above conventions as the *pure monad model.*

Consequently, we have the chronometrically invariant condition represented by

$$f\left(v^i, dt\right) = 0\,, \tag{3.1}$$

$$f\left(v_i, \frac{\partial}{\partial t}\right) \neq 0\,. \tag{3.2}$$

Therefore, with respect to the material hypersurface $\mathrm{C}^3(t)$, we see that

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}} = -\,g_{ik} + \frac{1}{c^2}\,v_i v_k\,, \tag{3.3}$$

$$h^{ik} = -\,g^{ik}, \qquad h^i_\alpha = -\,\delta^i_\alpha\,, \tag{3.4}$$

$$v_i = -\,c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \qquad v^i = -\,c\sqrt{g_{00}}\;g^{0i}, \tag{3.5}$$

$$v^2 = v_i v^i = h_{ik}\,v^i v^k\,, \tag{3.6}$$

$$u_0 = \sqrt{g_{00}}\,, \qquad u^0 = \frac{1}{\sqrt{g_{00}}}\,, \tag{3.7}$$

$$u_i = \frac{g_{0i}}{\sqrt{g_{00}}} = -\frac{v_i}{c}\,, \qquad u^i = 0\,, \tag{3.8}$$

$$dY^i = dx^i. \tag{3.9}$$

In our theory, Zelmanov's usual differential operators of chronometricity are given by

$$\frac{\partial}{\partial Y^i} = \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\,v_i\,\frac{{}^*\partial}{\partial t}\,, \tag{3.10}$$

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \tag{3.11}$$

$$\frac{\partial}{\partial Y^i}\frac{{}^*\partial}{\partial t} - \frac{{}^*\partial}{\partial t}\frac{\partial}{\partial Y^i} = \frac{{}^*\partial^2}{\partial x^i\,\partial t} - \frac{{}^*\partial^2}{\partial t\,\partial x^i} = \frac{1}{c^2}\,F_i\,\frac{{}^*\partial}{\partial t}\,, \tag{3.12}$$

$$\frac{\partial^2}{\partial Y^k\,\partial Y^i} - \frac{\partial^2}{\partial Y^i\partial Y^k} = \frac{{}^*\partial^2}{\partial x^k\,\partial x^i} - \frac{{}^*\partial^2}{\partial x^i\,\partial x^k} = -\frac{2}{c^2}\,A_{ik}\,\frac{{}^*\partial}{\partial t}\,. \tag{3.13}$$

Here the three-dimensional gravitational-inertial force, material deformation, and angular momentum are simply given by the three-dimensional chronometrically invariant components of the four-dimensional quantities F_α, $D_{\alpha\beta}$, and $A_{\alpha\beta}$ of the preceding §2 — in the case of vanishing background torsion — as follows:

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \tag{3.14}$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \qquad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \tag{3.15}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{3.16}$$

where the associated Zelmanov identities are

$$\frac{\partial A_{ik}}{\partial Y^l} + \frac{\partial A_{kl}}{\partial Y^i} + \frac{\partial A_{li}}{\partial Y^k} = \frac{^*\partial A_{ik}}{\partial x^l} + \frac{^*\partial A_{kl}}{\partial x^i} + \frac{^*\partial A_{li}}{\partial x^k} = $$
$$= -\frac{1}{c^2}\left(A_{ik}F_l + A_{kl}F_i + A_{li}F_k\right), \tag{3.17}$$

$$\frac{^*\partial A_{ik}}{\partial t} = \frac{1}{2}\left(\frac{\partial F_i}{\partial Y^k} - \frac{\partial F_k}{\partial Y^i}\right) = \frac{1}{2}\left(\frac{^*\partial F_i}{\partial x^k} - \frac{^*\partial F_k}{\partial x^i}\right), \tag{3.18}$$

and the gravitational potential scalar is

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right). \tag{3.19}$$

The symmetric chronometrically invariant connection of Zelmanov can be given here by

$$\Delta^i_{kl} = \Omega^i_{(kl)} + h^{ip}\left(h_{kq}\Omega^q_{[pl]} + h_{lq}\Omega^q_{[pk]}\right) = \frac{1}{2}\,h^{ip}\left(\frac{\partial h_{pk}}{\partial Y^l} - \frac{\partial h_{kl}}{\partial Y^p} + \frac{\partial h_{lp}}{\partial Y^k}\right) =$$
$$= \frac{1}{2}\,h^{ip}\left(\frac{^*\partial h_{pk}}{\partial x^l} - \frac{^*\partial h_{kl}}{\partial x^p} + \frac{^*\partial h_{lp}}{\partial x^k}\right) =$$
$$= \frac{1}{2}\,h^{ip}\left(\frac{\partial h_{pk}}{\partial x^l} - \frac{\partial h_{kl}}{\partial x^p} + \frac{\partial h_{lp}}{\partial x^k}\right) + \frac{1}{c^2}\left(D^i_k v_l - D_{kl}v^i + D^i_l v_k\right). \tag{3.20}$$

Note that while the extrinsic curvature tensor Z_{ik} is naturally asymmetric in our theory (in order to account for geometrized material vorticity), we might impose symmetry upon the material connection Ω^p_{ik} whenever convenient (or else we can associate its anti-symmetric part, through projection with respect to the background torsion, with the electromagnetic and chromodynamical gauge fields, as we have done, e.g., in [3] and [5]).

Now, with respect to the geometrized dynamical relations of the preceding section, we obtain

$$R_{\mu\nu\rho\sigma} = h^j_\nu h^k_\rho h^l_\sigma\left(h^i_\mu(R_{ijkl} - Z_{ik}Z_{jl} + Z_{il}Z_{jk}) + S_{\mu jkl} - u_\mu u^\lambda S_{\lambda jkl}\right) +$$
$$+ u_\mu X_{\nu\rho\sigma} - u_\nu X_{\mu\rho\sigma} + u_\rho Y_{\mu\nu\sigma} - u_\sigma Y_{\mu\nu\rho} + u_\mu u_\rho J_{\nu\sigma} - u_\mu u_\sigma J_{\nu\rho} +$$
$$+ u_\nu u_\rho K_{\mu\sigma} - u_\nu u_\sigma K_{\mu\rho}, \tag{3.21}$$

where

$$X_{\alpha\beta\gamma} = \frac{R_{0\alpha\beta\gamma}}{\sqrt{g_{00}}}, \qquad Y_{\alpha\beta\gamma} = \frac{R_{\alpha\beta0\gamma}}{\sqrt{g_{00}}}, \tag{3.22}$$

$$J_{\alpha\beta} = \frac{R_{0\alpha0\beta}}{g_{00}}, \qquad K_{\alpha\beta} = \frac{R_{\alpha00\beta}}{g_{00}} = -J_{\alpha\beta}. \tag{3.23}$$

These quantities, whose three-dimensional components may be linked with Zelmanov's various three-dimensional curvature tensors [1, 2], appear to correspond to certain generalized currents.

Further calculation reveals that

$$X_{\mu\nu\rho} + u_\nu J_{\mu\rho} - u_\rho J_{\mu\nu} = \\ = -\,h^i_\mu h^k_\nu h^l_\rho \left(\nabla_l\, Z_{ik} - \nabla_k\, Z_{il} + 2\,\Omega^{p}_{[kl]}\, Z_{ip} - S_{\lambda ikl}\, u^\lambda\right), \quad (3.24)$$

$$Y_{\mu\nu\rho} = \frac{2\sqrt{g_{00}}}{\sqrt{g_{00}} - 1}\left(u_\nu\, J_{\mu\rho} - u_\mu\, J_{\nu\rho}\right). \quad (3.25)$$

Therefore, the immediate general significance of these currents lies in the dynamical formation of matter itself with respect to the background structure of the world-geometry (represented by M^4).

§4. Hydrodynamical unification of physical fields. In this section, we shall deal with the explicit structure of the connection form underlying the world-manifold M^4, as well as that of matter — the material hypersurface $\mathrm{C}^3(t)$, by recalling certain fundamental aspects of our particular approach to the geometric unification of physical fields outlined in [3] and [5], which very naturally gives us the correct equation of motion for a particle (endowed with structure) moving in gravitational and electromagnetic fields while internally also experiencing the Yang-Mills gauge field, i.e., as an intrinsic geodesic equation of motion given by the following generalized metric-compatible connection form:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}(x,u) = \frac{1}{2}\, g^{\alpha\rho}\left(\frac{\partial g_{\rho\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho} + \frac{\partial g_{\gamma\rho}}{\partial x^\beta}\right) + \\ + \frac{e}{2mc^2}\left(F_{\beta\gamma}u^\alpha - F^{\alpha\cdot}_{\cdot\beta}u_\gamma - F^{\alpha\cdot}_{\cdot\gamma}u_\beta\right) + S^{\alpha\cdot\cdot}_{\cdot\beta\gamma} - g^{\alpha\rho}\left(S_{\beta\rho\gamma} + S_{\gamma\rho\beta}\right). \quad (4.1)$$

The anti-symmetric electromagnetic field tensor $F_{\alpha\beta}$ is fully geometrized through the relation

$$F_{\alpha\beta} = 2\,\frac{mc^2}{e}\,\Gamma^\lambda_{[\alpha\beta]}\, u_\lambda\,, \quad (4.2)$$

whose interior structure is given by the geometrized Yang-Mills gauge field [5], here in terms of the *internal* material coordinates of $\mathrm{C}^3(t)$ as

$$F^i_{\alpha\beta} = -\,2\,h^i_\lambda\Gamma^\lambda_{[\alpha\beta]} = \frac{\partial A^i_\alpha}{\partial x^\beta} - \frac{\partial A^i_\beta}{\partial x^\alpha} + i\,\hat{g}\,\epsilon^{i\cdot\cdot}_{\cdot kl}\, A^k_\alpha A^l_\beta + 2\,Z^{i\cdot}_{\cdot k}\, A^k_{[\alpha}\, u_{\beta]}\,, \quad (4.3)$$

$$F_{\alpha\beta} = \frac{mc^2}{e}\, F^i_{\alpha\beta}\, u_i\,, \qquad \Omega^i_{[kl]} = \frac{1}{2}\, i\,\hat{g}\,\epsilon^{i\cdot\cdot}_{\cdot kl}\,, \quad (4.4)$$

where $A^i_\alpha = -h^i_\alpha$ is the gauge field strength (not to be confused with the angular momentum), $\hat{g}$ is a coupling constant, and $\epsilon^{i\cdot\cdot}_{\cdot kl}$ is the three-dimensional permutation tensor.

The material spin tensor $S^{\alpha\cdot\cdot}_{\cdot\beta\gamma}$ is readily identified here through the anti-symmetric part of the four-dimensional form of the extrinsic curvature $\varphi_{\alpha\beta}$ (i.e., the material vorticity $\omega_{\alpha\beta}$) of M^4:

$$S^{\alpha\cdot\cdot}_{\cdot\beta\gamma} = S^{\alpha\cdot}_{\cdot\beta}\, u_\gamma - S^{\alpha\cdot}_{\cdot\gamma}\, u_\beta \,, \tag{4.5}$$

$$S_{\alpha\beta} = \hat{s}\,\varphi_{[\alpha\beta]} = \hat{s}\,\omega_{\alpha\beta} = \frac{1}{2}\,\hat{s}\left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta\right), \tag{4.6}$$

where $\hat{s}$ is a constant spin coefficient, which can possibly be linked to the electric charge e, the mass m, and the speed of light in vacuum c, and hence to the Planck-Dirac constant $\hbar$ as well, such that we can express the connection form more compactly as

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\rho}\left(\frac{\partial g_{\rho\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho} + \frac{\partial g_{\gamma\rho}}{\partial x^\beta}\right) + \\ + \frac{e}{2mc^2}\left(F_{\beta\gamma} u^\alpha - F^{\alpha\cdot}_{\cdot\beta}\, u_\gamma - F^{\alpha\cdot}_{\cdot\gamma}\, u_\beta\right) + 2S^{\alpha\cdot}_{\cdot\beta}\, u_\gamma \,. \tag{4.7}$$

Therefore, owing to the fully intrinsic dynamics of the geometrized physical fields located in M^4, i.e.,

$$\frac{Du^\alpha}{ds} = u^\beta\, \nabla_\beta\, u^\alpha = 0 \,, \tag{4.8}$$

we see that the following condition is naturally satisfied:

$$S_{\alpha\beta}\, u^\beta = 0 \tag{4.9}$$

in addition to the equation of motion

$$mc^2\left(\frac{du^\alpha}{ds} + \Delta^\alpha_{\beta\gamma}\, u^\beta u^\gamma\right) = eF^{\alpha\cdot}_{\cdot\beta}\, u^\beta, \tag{4.10}$$

where the usual connection coefficients are

$$\Delta^\alpha_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\rho}\left(\frac{\partial g_{\rho\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho} + \frac{\partial g_{\gamma\rho}}{\partial x^\beta}\right). \tag{4.11}$$

Meanwhile, from §2, we note that

$$\varphi_{\alpha\beta} = \nabla_\beta\, u_\alpha = -\,h^i_\alpha\, h^k_\beta\, Z_{ik} \,, \tag{4.12}$$

$$Z_{ik} = -\,u_\alpha\, \nabla_k\, h^\alpha_i \,, \tag{4.13}$$

$$\varphi_{\alpha\beta}\, u^\alpha = 0 \,, \qquad \varphi_{\alpha\beta}\, u^\beta = 0 \,, \tag{4.14}$$

and so we (re-)obtain

$$\Omega^p_{ik} = h^p_\alpha \frac{\partial h^\alpha_i}{\partial Y^k} - h^p_\alpha \Gamma^\alpha_{\beta\gamma} h^\beta_i h^\gamma_k \,, \tag{4.15}$$

$$\Gamma^\alpha_{\beta\gamma} = h^\alpha_i \frac{\partial h^i_\beta}{\partial x^\gamma} - h^\alpha_p \Omega^p_{ik} h^i_\beta h^k_\gamma - \varphi_{\beta\gamma} u^\alpha + u^\alpha \frac{\partial u_\beta}{\partial x^\gamma} + \varphi^{\alpha\cdot}_{\cdot\gamma} u_\beta \,, \tag{4.16}$$

where the material connection of $C^3(t)$ can be explicitly expressed as

$$\Omega^i_{kl} = \frac{1}{2}\, h^{ip}\left(\frac{\partial h_{pk}}{\partial Y^l} - \frac{\partial h_{kl}}{\partial Y^p} + \frac{\partial h_{lp}}{\partial Y^k}\right) + \frac{1}{2}\, i\,\hat{g}\,\epsilon^{i\cdot\cdot}_{\cdot kl} \,, \tag{4.17}$$

or, in other words,

$$\Omega^i_{kl} = \frac{1}{2}\, h^{ip}\left(\frac{\partial h_{pk}}{\partial x^l} - \frac{\partial h_{kl}}{\partial x^p} + \frac{\partial h_{lp}}{\partial x^k}\right) + \\ + \frac{1}{c^2}\left(D^i_k v_l - D_{kl} v^i + D^i_l v_k\right) + \frac{1}{2}\, i\,\hat{g}\,\epsilon^{i\cdot\cdot}_{\cdot kl} \,. \tag{4.18}$$

This way, we have also obtained the fundamental structural forms corresponding to the immediate structure of our geometric theory of chiral elasticity [6], which, to a certain extent, is capable of encompassing the elastodynamics of matter in our present theory, as represented by the material hypersurface $C^3(t)$.

§5. A Machian monad model of the Universe. We shall now turn towards developing a particular pure monad model, i.e., one in which the Universe possesses absolute angular momentum such that matter arises entirely from the intrinsic inhomogeneity and anisotropy emerging from the non-orientability and discontinuity of the very geometry of the material hypersurface $C^3(t)$ with respect to the background space-time M^4. This goes down to saying that the cosmos has neither "inside" nor "outside" as graphically outlined in [4], and that each point in space-time indeed possesses *intrinsic informational spin*, irrespective of whether or not its corresponding empirical constitution possesses extrinsic angular momentum.

Recall, from the previous section, that the anti-symmetric part of the material connection form is given by the complex expression

$$\Omega^i_{[kl]} = \frac{1}{2}\, i\,\hat{g}\,\epsilon^{i\cdot\cdot}_{\cdot kl} \,, \tag{5.1}$$

which displays the internal constitution of matter in terms of the gauge coupling constant $\hat{g}$. Now, the four-dimensional permutation tensor is readily given by

$$\epsilon_{ikl}\, u_\gamma = -\,\epsilon_{\alpha\beta\rho\gamma}\, h_i^\alpha\, h_k^\beta h_l^\rho\,, \tag{5.2}$$

i.e.,

$$\epsilon_{\alpha\beta\rho\gamma} = -\,\epsilon_{ikl}\, h_\alpha^i h_\beta^k h_\rho^l\, u_\gamma + a_{\alpha\beta\rho\gamma} + b_{\alpha\beta\rho\gamma} + c_{\alpha\beta\rho\gamma}\,, \tag{5.3}$$

$$\epsilon_{ikl} = -\,\epsilon_{\alpha\beta\rho\gamma}\, h_i^\alpha\, h_k^\beta h_l^\rho\, u^\gamma, \tag{5.4}$$

$$a_{\alpha\beta\rho\gamma} = \epsilon_{\mu\beta\rho\gamma}\, u^\mu u_\alpha\,, \tag{5.5}$$

$$b_{\alpha\beta\rho\gamma} = \epsilon_{\alpha\mu\rho\gamma}\, u^\mu u_\beta\,, \tag{5.6}$$

$$c_{\alpha\beta\rho\gamma} = \epsilon_{\alpha\beta\mu\gamma}\, u^\mu u_\rho\,. \tag{5.7}$$

We therefore see that

$$\Omega^i_{[kl]} = -\frac{1}{2}\, i\,\hat{g}\,\epsilon^{\alpha\cdot\cdot\cdot}_{\cdot\beta\rho\gamma}\, h_\alpha^i\, h_k^\beta h_l^\rho\, u^\gamma, \tag{5.8}$$

and, in particular, that

$$\nabla_m\, \Omega^i_{[kl]} = -\frac{1}{2}\, i\,\hat{g}\, h_\alpha^i\, h_k^\beta\, h_l^\rho\, h_m^\lambda\, \epsilon^{\alpha\cdot\cdot\cdot}_{\cdot\beta\rho\gamma}\, \varphi^{\gamma\cdot}_{\cdot\lambda}\,. \tag{5.9}$$

The spatial curvature giving rise to matter can now be written as

$$R^{i\cdot\cdot\cdot}_{\cdot jkl} = B^{i\cdot\cdot\cdot}_{\cdot jkl} + M^{i\cdot\cdot\cdot}_{\cdot jkl}\,, \tag{5.10}$$

$$B^{i\cdot\cdot\cdot}_{\cdot jkl} = \frac{\partial P^i_{jl}}{\partial Y^k} - \frac{\partial P^i_{jk}}{\partial Y^l} + P^m_{jl}\, P^i_{mk} - P^m_{jk}\, P^i_{ml}\,, \tag{5.11}$$

$$M^{i\cdot\cdot\cdot}_{\cdot jkl} = \hat{\nabla}_k\, C^i_{jl} - \hat{\nabla}_l\, C^i_{jk} + C^m_{jl}\, C^i_{mk} - C^m_{jk}\, C^i_{ml}\,, \tag{5.12}$$

$$P^i_{kl} = \frac{1}{2}\, h^{ip}\left(\frac{\partial h_{pk}}{\partial Y^l} - \frac{\partial h_{kl}}{\partial Y^p} + \frac{\partial h_{lp}}{\partial Y^k}\right) = \Delta^i_{kl}\,, \tag{5.13}$$

$$C^i_{kl} = \Omega^i_{[kl]} - h^{ip}\left(h_{km}\, \Omega^m_{[pl]} + h_{lm}\, \Omega^m_{[pk]}\right) = \Omega^i_{[kl]}\,, \tag{5.14}$$

where $\hat{\nabla}$ denotes covariant differentiation with respect to the symmetric connection form $P^i_{kl}(\Delta^i_{kl})$.

The special integrability conditions for our particular model of space-time geometry possessing absolute angular momentum will be given by

$$h_i^\beta\, \Delta^\alpha_{\beta\gamma} = 0\,, \tag{5.15}$$

$$h_i^\alpha\, h_j^\beta\, h_k^\rho\, h_l^\gamma\left(R_{\alpha\beta\rho\gamma} + \varphi_{\alpha\rho}\, \varphi_{\beta\gamma} - \varphi_{\alpha\gamma}\, \varphi_{\beta\rho}\right) = 0\,, \tag{5.16}$$

such that, explicitly,

$$\Omega^i_{kl} = h_\alpha^i\, \frac{\partial h_k^\alpha}{\partial Y^l} = h_\alpha^i\, \frac{{}^*\partial h_k^\alpha}{\partial x^l} = h_\alpha^i\left(\frac{\partial h_k^\alpha}{\partial x^l} + \frac{1}{c^2}\, v_l\, \frac{{}^*\partial h_k^\alpha}{\partial t}\right). \tag{5.17}$$

We therefore obtain

$$\Delta^{\alpha}_{\beta\gamma} = \frac{1}{2}\,u_{\beta}\,g^{\alpha\rho}\left(\frac{\partial u_{\rho}}{\partial x^{\gamma}} - \frac{\partial u_{\gamma}}{\partial x^{\rho}} + \frac{dg_{\rho\gamma}}{ds}\right) + \frac{1}{2}\,u_{\beta}\left(g_{\rho\gamma}\,\frac{\partial u^{\rho}}{\partial x_{\alpha}} - \frac{\partial u^{\alpha}}{\partial x^{\gamma}}\right), \quad (5.18)$$

i.e.,

$$\Delta^{\alpha}_{\beta\gamma} = \frac{1}{2}\,u_{\beta}\,g^{\alpha\rho}\,\frac{dg_{\rho\gamma}}{ds} + u_{\beta}\left(g^{\alpha\rho}\,\Gamma^{\sigma}_{[\rho\gamma]}\,u_{\sigma} + u^{\rho}\,\Gamma^{\alpha}_{[\rho\gamma]}\right) \quad (5.19)$$

such that the world-velocity u^{α} plays the role of a fundamental "metric vector".

This way, matter (material curvature), and hence inertia, arises purely from the segmental torsional (discontinuity) curvature as follows:

$$R^{i\cdots}_{\cdot jkl} = -\,h^{i}_{\alpha}\left[\frac{\partial}{\partial Y^{l}}\left(\frac{\partial h^{\alpha}_{j}}{\partial Y^{k}}\right) - \frac{\partial}{\partial Y^{k}}\left(\frac{\partial h^{\alpha}_{j}}{\partial Y^{l}}\right)\right], \quad (5.20)$$

i.e.,

$$R^{i\cdots}_{\cdot jkl} = -\frac{2}{c^{2}}\,h^{i}_{\alpha}\,A_{kl}\,\frac{{}^{*}\partial h^{\alpha}_{j}}{\partial t}\,, \quad (5.21)$$

where the angular momentum A_{ik} is given by

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_{k}}{\partial x^{i}} - \frac{\partial v_{i}}{\partial x^{k}}\right) + \frac{1}{2c^{2}}\left(F_{i}\,v_{k} - F_{k}\,v_{i}\right) - \\ - u_{\alpha}\left(c\,\Gamma^{\alpha}_{[ik]} + \frac{v_{i}}{\sqrt{g_{00}}}\,\Gamma^{\alpha}_{[0k]} + \frac{v_{k}}{\sqrt{g_{00}}}\,\Gamma^{\alpha}_{[i0]}\right), \quad (5.22)$$

$$F_{i} = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^{i}} - \frac{\partial v_{i}}{\partial t}\right) + 2\,\frac{c^{2}}{\sqrt{g_{00}}}\,\Gamma^{\alpha}_{[0i]}\,u_{\alpha}\,, \quad (5.23)$$

$$v_{i} = -\,c\,\frac{g_{0i}}{\sqrt{g_{00}}} = -\,c\,u_{i}\,, \qquad u^{i} = 0\,, \quad (5.24)$$

$$u_{0} = \sqrt{g_{00}}\,, \qquad u^{0} = \frac{1}{\sqrt{g_{00}}}\,, \quad (5.25)$$

$$\mathrm{w} = c^{2}\left(1 - \sqrt{g_{00}}\right). \quad (5.26)$$

In this particular scheme, therefore, every constitutive object in the Universe spins in the topological sense of gaining informational spin from the very formation of matter itself. Inertia would then be a property of matter directly arising from this intrinsic mechanism of spin, which encompasses the geometric formation of all massive objects at any scale. This, in turn, subtly corresponds to the Machian conjecture of the inertia (mass) of an object being dependent on a distant, massive frame of reference (if not all other massive objects in the Universe). However, since our peculiar geometric mechanism here exists at every point of space-time, and in the topological background of things, the corresponding generation of inertia is simply more intrinsic than the initial Machian scheme. Accordingly, there is no need to invoke the existence of a distant galactic frame of reference, other than the general non-orientability and curvature-generating discreteness of the hypersurface representing matter.

§6. Conclusion. We have outlined a seminal sketch of a fully hydrodynamical geometric theory of space-time and fields, which might complement Zelmanov's chronometric formulation of the General Theory of Relativity. In our theory, chronometricity is particularly geometrized through the unique hydrodynamical nature of the asymmetric extrinsic curvature of the material hypersurface.

Following our previous works we have unified the gravitational and electromagnetic fields, with chromodynamics arising from the fully geometrized inner structure of the electromagnetic field, which is shown to be the Yang-Mills gauge field (appearing here in its generalized form). In the present work, it is interesting to note that the role of the non-Abelian gauge field (represented by its components, namely, A^i_α) is very naturally played by the projective chronometric structure (with components h^i_α), and so the inner constitution of elementary particles cannot be divorced from chronometricity at all.

In our approach to Mach's principle through a pure monad model possessing absolute angular momentum, the unique Kleinian topology of the Universe gives rise to inertia in terms of the non-orientable spin dynamics and discrete intrinsic geometry of the material hypersurface, rendering the respective generation of inertia both local and global (i.e., signifying, in a cosmological sense, scale-independence as well as intrinsic topological interdependence among "particulars" and "universals").

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This journal is named after Abraham Zelmanov (1913–1987), a prominent scientist working in the General Theory of Relativity and cosmology, whose main goal was the mathematical apparatus for calculation of the physical observable quantities in the General Theory of Relativity (it is also known as the theory of chronometric invariants). He also developed the basics of the theory of an inhomogeneous anisotropic universe, and the classification of all possible models of cosmology which could be theoretically conceivable in the space-time of General Relativity (the Zelmanov classification). He also introduced the Anthropic Principle and the Infinite Relativity Principle in cosmology.

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