Lichnérowicz's Theory of Spinors in General Relativity: the Zelmanov Approach

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Abstract: In this paper, we apply Abraham Zelmanov's theory of chronometric invariants to the spinor formalism, based on Lichnérowicz's initial spinor formalism extended to the General Theory of Relativity. From the classical theory, we make use of the Dirac current which is shown to be a real four-vector, and by an appropriate choice of the compatible gamma matrices, this current retains all the properties of a space-time vector. Its components are uniquely expressed in terms of spinor components, and we eventually obtain the desired physically observable spinors in the sense of Zelmanov, next to the scalar, vector, and tensor quantities.

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Introduction. Preliminary conventions:

- $\hbar = 1$ and $c \neq 1$;
- Four-dimensional general basis e_μ and numbering of the four 4×4 Dirac gamma matrices with Greek indices: $\mu, \nu = 0, 1, 2, 3$;
- Three-dimensional general basis e_α with Greek indices: $\alpha, \beta = 1, 2, 3$;
- Four-dimensional coordinates with Latin indices: $a, b, c, \ldots, f = 0, 1, 2, 3$;
- Three-dimensional coordinates with Latin indices: $i, j, k, l, m, \ldots = 1, 2, 3$;
- Spinor indices with capital Latin indices: $A, B = 1, 2, 3, 4$.

We first briefly recall the essence of Zelmanov's theory: the dynamic fundamental observer can be coupled with his physical referential system whose general space-time possesses a gravitational field, generally subject to rotation and deformation. Physical-mathematical quantities (scalars, vectors, tensors) as measured in the observer's accompanying frame of reference, are called *physically observable quantities* if and only if they result *uniquely* from the *chronometric projection* of the generally covariant four-dimensional quantities onto the time line and the spatial section in such a way that the new semi-three-dimensional quantities depend everywhere on the monad vector (world-velocity): those are known as *chronometrically invariant quantities.*

If the spatial sections are everywhere orthogonal to the time line, the enveloping space is said to be *holonomic.* In general, the real space-time (e.g. of the Metagalaxy) is *non-homogeneous* and *non-isotropic*, i.e. it is *non-holonomic.*

Besides vectors and tensors, we intend here to derive a law setting Zelmanov's physically observable properties for another type of quantities, namely *spinors.* To achieve this, we will first follow the Lichnérowicz analysis which formally defines spinors within General Relativity, and which leads to the well-known Dirac equation for the electron in a pseudo-Riemannian space-time. We will then infer the *Dirac current* from the *spinor Lagrangian*, which is shown to retain all properties of a real four-vector.

With a special choice of the gamma matrices compatible with the regular spinor theory [1], we define the Dirac current as a space-time vector whose components are exclusively expressed in terms of spinors.

These spinors, as they are physically observed, are thus modified through the *chronometric properties* of the general space, i.e. the *linear* *velocity of space rotation* v_i, and the *gravitational force* F_i, as well as the gravitational potential w.

Three-dimensional *non-holonomity* and *deformation* of space, which are respectively represented by the antisymmetric and symmetric chronometrically invariant tensors A_{ik}, and D_{ik}, appear in the Christoffel symbols' components formulated uniquely in terms of physically observable quantities [2], which are to be part of the Riemann spinor connection.

Accordingly, we can construct a *Dirac-Zelmanov equation* for the electron interacting with a four-potential A^{μ}, whose chronometrically invariant projections (physically observable components) only apply here to A_0.

It is essential to note that the inferred Dirac-Zelmanov equation does also comply with the positron equation, as easily shown below.

§1. The Riemannian spinor field

§1.1. General Riemannian space-time. Let V_4 be a C^{∞}-differentiable Riemann four-manifold which admits a structural group, namely the *homogeneous* (or *full*) *Lorentz group* denoted by $\mathbf{L}(4)$.

The metric is locally written on an open neighbourhood of the manifold V_4 as

$$ds^2 = g_{ab}\, dx^a dx^b, \tag{1}$$

which is equivalent to writing

$$ds^2 = \eta_{\mu\nu}\, \theta^{\mu}\theta^{\nu}, \tag{2}$$

where $\eta_{\mu\nu}$ is the *Minkowski metric tensor*: $(1, -1, -1, -1)$.

The θ^{μ} are the four *Pfaffian forms* (see a formal definition of these in Appendix) which are related to the coordinate bases by

$$\theta^{\mu} = \mathbf{a}^{\mu}_{a}\, dx^a, \tag{3}$$

where the $\mathbf{a}^{\mu}_{a}$ form the *tetrad part* that carries the curved space-time properties.

We systematically refer V_4 to orthonormal bases, which are the elements of a fiber space $E(V_4)$, whose structural group is the full Lorentz group $\mathbf{L}(4)$.

Let now $\mathbf{F}$ be a matrix $\mathbf{F} \in \mathbf{L}(4)$. The *global orientation* of the manifold V_4 is a *pseudoscalar* denoted by $\mathbf{o}$, whose square is 1, and which is defined, with respect to the system of frames y of $E(V_4)$, as the component $\mathbf{o}_y = \pm 1$, such that if $y = y'\,\mathbf{F}$, we get $\mathbf{o}_y = \mathbf{o}_{y'}\mathbf{o}_{\mathbf{F}}$.

We next define the antisymmetric tensor $\varepsilon^{a_1\ldots a_0}_{b_1\ldots b_0}$ as follows: its components are $+1$ if the upper indices' series is an even permutation of the series of lower indices all assumed distinct, and -1 for an odd permutation, and 0 otherwise. The covariant derivative of this tensor is zero. To simplify the notation, we may just write $\varepsilon^{a_1\ldots a_0}_{1\ldots 0} = \varepsilon^{a_1\ldots a_0}$, $\varepsilon^{1\ldots 0}_{a_1\ldots a_0} = \varepsilon_{a_1\ldots a_0}$.

If V_4 is oriented, we set the orientable volume element tensor η (whose covariant derivative is also zero), as

$$\eta_{abcd} = \sqrt{-g}\;\varepsilon_{abcd}\,, \qquad \eta^{abcd} = \frac{1}{\sqrt{-g}}\,\varepsilon^{abcd}.$$

In a positively-oriented basis, the Levi-Civita tensor $\boldsymbol{\varepsilon}$ would have components defined by $\varepsilon_{0123} = 1$. This amounts to a choice of orientation of the four-dimensional manifold in where the orthonormal basis e_μ represents a Lorentz frame with e_0 pointing toward the future and e_δ being a *right-handed triad*.

As a result, for two global orientations $\mathbf{o}$ and $\mathbf{o}'$ on V_4, we have at most two total orientations $\mathbf{o}$ and $-\mathbf{o}$, which define the *orientable* volume element $\boldsymbol{\eta}$ as

$$\mathbf{o}\,\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Now, on V_4, we define the *temporal orientation* or *time orientation* $\mathbf{t}$ with respect to the set $y \in \mathrm{E}(\mathrm{V}_4)$, by one component $\mathbf{t}_y = \pm 1$, such that if $y = y'\mathbf{F}$, we have

$$\mathbf{t}_y = \mathbf{t}_{y'}\mathbf{t}_{\mathbf{F}}$$

where V_4 admits *at most* two time orientations $\mathbf{t}$ and $-\mathbf{t}$.

Any vector e_0 ($e_0^2 = 1$) at x is oriented towards the *future* (respectively the *past*), when the components of $\mathbf{t}$ with respect to the orthonormal frames (e_0, e_δ) is 1 (respectively -1).

§1.2. The gamma matrices. In what follows, Λ^* is the complex conjugate of an arbitrary matrix Λ, Λ^{T} is the transpose of Λ, while $\widetilde{\Lambda}$ is the classical adjoint of Λ.

Let $\Re$ be the real numbers set on which the vector space is defined. This vector space is spanned by the 16 matrices

$$\mathbf{I}, \quad \gamma_\mu, \quad \gamma_\mu\gamma_\nu\,, \quad \gamma_\mu\gamma_\nu\gamma_\alpha\,, \quad \gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\,, \tag{4}$$

where $\mathbf{I}$ is the *unit matrix*, and the four *gamma matrices* are denoted by

$$\gamma^\mu \equiv \gamma^{\mu A}_{B}\,. \tag{5}$$

Note that all indices are distinct here.

With the tensor $\eta_{\mu\nu}$, we write the fundamental relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\,\eta_{\mu\nu}\,\mathbf{I}, \tag{6}$$

which is verified by the gamma matrices, with the following elements

$$\left.\begin{aligned}
\gamma^0 = \gamma_0 = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\\
-\gamma^1 = \gamma_1 = i\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\\
-\gamma^2 = \gamma_2 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\\
-\gamma^3 = \gamma_3 = i\begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}
\end{aligned}\right\}. \tag{7}$$

This system of matrices is also called a *standard* representation, as opposed, for example, to the Majorana representation [3, p. 108], or to the usual spinorial representation.

We note that

$$\widetilde{\gamma}_\mu = -\eta^{\mu\mu}\gamma_\mu \tag{8}$$

and also

$$\det(\gamma_\mu) = 1\,, \quad \mathrm{tr}\,(\gamma_\mu) = 0\,, \quad \mathrm{tr}\,(\gamma_\mu\gamma_\nu) = 0\,, \quad \mathrm{tr}\,(\gamma_\mu\gamma_\nu\gamma_\sigma) = 0$$

for $\mu \neq \nu \neq \sigma$, etc.

§2. The spinor concept

§2.1. The isomorphism p. We shortly recall the definition of the spin group Spin(4) which is said to be the *covering* of $\mathbf{L}(4)$. The projection $\mathbf{p}$ of Spin(4) onto $\mathbf{L}(4)$ is such that if

$$\mathbf{A} = (A^{\mu'}_{\nu}) = \mathbf{p}\Lambda\,, \quad \mathbf{A} \in \mathbf{L}(4)\,, \quad \Lambda \in \mathrm{Spin}(4)$$

we must have

$$\Lambda\gamma_\mu\Lambda^{-1} = A_\mu^{\nu'}\gamma_{\nu'} \,,$$

where **p** is regarded as the *isomorphism* of the spin group **S**(4) on the Lorentz group **L**(4).

In view of expressing the spinors as physically observable quantities, one could start with the regular Riemannian line-element

$$ds^2 = g_{ab}\,dx^a dx^b,$$

which becomes in Zelmanov's theory

$$ds = c^2 d\tau^2 - d\sigma^2,$$

where

$$d\tau = \sqrt{g_{00}}\,dt + \frac{1}{c}\,\frac{g_{0i}\,dx^i}{\sqrt{g_{00}}}\,,$$

$$d\sigma^2 = h_{ik}\,dx^i dx^k$$

and

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\,g_{0k}}{g_{00}}\,.$$

(See [4], formula 1.29, in accordance with formula 84.6 of §84 [5].)

Hence considering $h^i_\mu h^\mu_k = \delta^i_k$, one could then start by writing the physically observable Dirac matrices as simply verifying the relation

$$\gamma_i\gamma_k + \gamma_k\gamma_i = -\,2\,\delta_{ik}\,\mathbf{I}$$

but the matrix $\gamma_0(x)$ is omitted and the isomorphism **p** can no longer apply.

Therefore, writing the *Pfaffian* interval as $ds^2 = \eta_{\mu\nu}\,\theta^\mu\theta^\nu$ allows us to preserve this isomorphism, which is the fundamental feature of any relativistic spinor theory. Like in the classical treatment, we thus maintain the relation (6), so that the gamma matrices are kept non-local. Proceeding with the Lichnérowicz formalism, we shall see that there is another way to obtain the spinors as unique observable quantities within Zelmanov's theory.

§2.2. Spinor definitions. From E(V_4), we define a principal *fiber space* denoted by S(V_4) whose each point z represents a general *spinor frame*: at this stage, it is essential to understand that Spin(4) is here its structural group. V_4 is therefore a vector space of 4×4 matrices with complex elements, which is acted upon by the Spin(4) group [6].

Let us denote by π the *canonical projection* of $\mathrm{S}(\mathrm{V}_4)$ onto V_4, and $\mathbf{p}$ the projection of $\mathrm{S}(\mathrm{V}_4)$ onto $\mathrm{E}(\mathrm{V}_4)$, so that a tensor of V_4 is referred to its frame by $y = \mathbf{p}\, z$.

The *contravariant 1-spinor* ψ at $x \in \mathrm{V}_4$ is defined as a mapping $z \to \psi(z)$ of $\pi^{-1}(x)$ onto V_4.

The *covariant 1-spinor* ϕ at x is a mapping $z \to \phi(z)$ of $\pi^{-1}(x)$ onto the space V_4, *dual* to V_4.

The contravariant 1-spinor ψ forms a vector space $\mathrm{S}x$ on the complex numbers, whereas the covariant 1-spinor forms the vector space $\mathrm{S}'x$ *dual* to $\mathrm{S}x$. The contravariant 1-spinor ψ has also its covariant 1-spinor counterpart expressed by

$$\bar{\psi} = \mathbf{t}\,\widetilde{\psi}\,\beta\,, \tag{9}$$

which is classically known as the *Dirac conjugate*, also expressed by

$$\bar{\psi} = \psi^{*}\gamma^{0} \tag{10}$$

with $\mathbf{t} = +1$. Herein, β is a matrix defined below in (12).

Conversely, any covariant 1-spinor ϕ is now the image of the contravariant 1-spinor $\mathbf{t}\beta\phi$.

§2.3. The charge conjugation and the adjoint operation. An antilinear mapping $\mathbf{C}$ of $\mathrm{S}x$ onto itself exists. It maps a 1-spinor ψ to another 1-spinor such as

$$\mathbf{C} : \psi \longrightarrow \psi\,\mathbf{C} = \psi^{*}.$$

We readily see that $\mathbf{C}^2 = \text{Identity}\,(\psi \to \psi)$, while $\mathbf{C}$ is known as the *charge conjugate operation*.

In particular, the charge conjugate of the covariant 1-spinor ϕ is

$$\mathbf{C}\phi = \psi^{*}$$

hence

$$(\mathbf{C}\phi, \psi) = (\phi, \mathbf{C}\psi)^{*}.$$

The relation (6) results from the identity $(u^{\mu}\gamma_{\mu})^2 = -(u^{\mu}u^{\nu}\eta_{\mu\nu})\mathbf{I}$, where $u^{\mu} \in \mathcal{C}$ (complex numbers), thus from (8) we find

$$\gamma_0\,\gamma_{\mu}\,\gamma_0^{-1} = -\widetilde{\gamma}_{\mu}\,. \tag{11}$$

Introducing now the real matrix $\beta = i\,\gamma_0$ which verifies $\beta^2 = \mathbf{I}$, the important relation can be derived from (10)

$$\beta\,\gamma_{\mu}\,\beta^{-1} = -\widetilde{\gamma}_{\mu}\,. \tag{12}$$

By defining an *antilinear mapping* $\mathbf{A}$ of $\mathrm{S}x$ onto $\mathrm{S}'x$ as

$$\mathbf{A} : \psi \longrightarrow \bar{\psi} = \mathbf{t}\,\widetilde{\psi}\,\beta \tag{13}$$

we have the *Dirac adjoint operation* $\mathbf{A}$.

We now consider a contravariant 1-spinor ψ which satisfies $\mathbf{C}\,\psi = \psi^*$. Thus

$$\mathbf{AC}\,\psi = \mathbf{t}\,\psi^{\mathrm{T}}\beta\,.$$

On the other hand, $\mathbf{A}\psi = \mathbf{t}\,\widetilde{\psi}\beta$, from which we infer

$$\mathbf{CA}\psi = \mathbf{t}\,\psi^{\mathrm{T}},$$

i.e. $\mathbf{CA}\,\psi = -\,\mathbf{t}\,\psi^{\mathrm{T}}\beta$. Therefore

$$\mathbf{AC}\,\psi = -\,\mathbf{CA}\,\psi\,.$$

The Dirac adjoint operator and the charge conjugation *anticommute* on the 1-spinors [7].

§3. The Riemannian Dirac equation

§3.1. The spinor connection. In order to write the Schrödinger equation under a relativistic form, Dirac introduced a four-component wave function ψ_A (see [8] and [9, p. 252]) expressed with the gamma matrices. In the classical theory, the expression $\gamma^{\mu A}_{B}\partial_\mu$ is known as the *Dirac operator*, and it is customary to omit the *spinor indices* A, B by simply writing $\gamma^{A}_{\mu B}$ so as to get $\gamma^\mu\partial_\mu$.

In a Riemannian situation, the derivative ∂_μ becomes D_μ with a Riemannian spinor connection defined as follows:

$$\boldsymbol{N} = -\,\frac{1}{4}\,\Gamma^{\mu\nu}\gamma_\mu\gamma_\nu = -\,\frac{1}{4}\,\Gamma^{\mu}_{\nu}\gamma_\mu\gamma^\nu. \tag{14}$$

Within a neighbourhood $\mathfrak{U}$ of $\mathrm{E}(\mathrm{V}_4)$, we define a connection 1-form $\boldsymbol{\Gamma}$ that is represented by either of the two matrices Γ^μ_ν or $\Gamma^{\mu\nu}$, and whose elements are linear forms.

The matrix $\boldsymbol{N}$ defines the *spinor connection* corresponding to $\boldsymbol{\Gamma}$. The elements of $\boldsymbol{N}$ are given by the local 1-forms

$$N^A_B = -\,\frac{1}{4}\,\Gamma^\mu_\nu\,\gamma^A_{\nu C}\,\gamma^{\nu C}_{B}\,.$$

By means of the Riemannian connection $\Gamma^\mu_{\nu\alpha}$ with respect to the frames in $\mathfrak{U}$, the corresponding coefficients of $\boldsymbol{N}$ are written

$$N^A_{B\alpha} = -\,\frac{1}{4}\,\Gamma^\mu_{\nu\alpha}\,\gamma^A_{\mu C}\,\gamma^{\nu C}_{B}\,. \tag{15}$$

§3.2. The Riemannian Dirac operators and subsequent Dirac equation. Some inspection shows that the absolute differential of the gamma matrices is given by

$$\mathrm{D}\,\gamma^{\mu} = d\gamma^{\mu} + \Gamma^{\mu}_{\nu}\gamma^{\nu} + (\boldsymbol{N}\gamma^{\mu} - \gamma^{\mu}\boldsymbol{N}) \tag{16}$$

and this differential is shown to be zero. With (15) it can also be inferred that the covariant derivative of a spinor ψ^{A} is

$$\mathrm{D}_{\mu}\psi^{A} = \partial_{\mu}\psi^{A} + N^{A}_{B\mu}\psi^{B} \tag{17}$$

and for the covariant 1-spinor ϕ

$$\mathrm{D}_{\nu}\,\phi_{A} = \partial_{\nu}\psi_{A} - N^{B}_{A\nu}\,\psi_{B}\,.$$

Introducing now the *Riemannian Dirac operators* W and $\bar{\mathrm{W}}$ as

$$\mathrm{W}\psi = \gamma^{\mu}\mathrm{D}_{\mu}\psi\,, \qquad \bar{\mathrm{W}}\phi = -\,\mathrm{D}_{\mu}\,\phi\gamma^{\mu} \tag{18}$$

for a massive spin-$\frac{1}{2}$-field, the *Riemannian Dirac equations* [10] are written as

$$(\mathrm{W} - m_0 c)\,\psi = 0\,, \tag{19}$$

$$\big(\bar{\mathrm{W}} - m_0 c\big)\,\phi = 0\,, \tag{20}$$

where the rest mass m_0 is usually attributed to the associated particle.

Classically, the Dirac massive equation is always written with the contravariant 1-spinor ψ, satisfying the *free field equation* (19) (no external interacting field).

In accordance with our previous results, the Dirac adjoint $\bar{\psi}$ thus satisfies

$$\big(\bar{\mathrm{W}} - m_0 c\big)\,\bar{\psi} = 0\,. \tag{21}$$

§3.3. The Dirac current vector density. In order to obtain the physically observable spinor quantities we are aiming at, we first define a space-time vector which is entirely expressed through the spinor formulation. For this purpose, we will rely on the Dirac current vector which is formally inferred from the Dirac Lagrangian of a massive fermion field. Such a Lagrangian is shown to be [11]

$$\mathcal{L}_{\mathrm{D}} = \frac{1}{2}\left[\,\bar{\psi}\gamma^{\mu}\mathrm{D}_{\mu}\psi - (\mathrm{D}_{\mu}\bar{\psi})\,\gamma^{\mu}\psi\,\right] - m_0 c\,\bar{\psi}\psi\,. \tag{22}$$

An alternative formula is given by

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}\,(\gamma^{\mu} - m_0 c)\,\psi\,.$$

Since these forms differ only by a divergence which vanishes at infinity, they generate the same action and correspond to the same physics.

Following Noether's theorem, we now apply the invariance rule to $\mathcal{L}_{\mathrm{D}}$ (22) upon the *global* transformations (where U is a positive scalar).

$$\psi \longrightarrow e^{iU}\psi, \qquad \bar{\psi} \longrightarrow \bar{\psi}\, e^{-iU}$$

for linear transformations of ψ, the respective Lagrangian variation is

$$\delta\mathcal{L}_{\mathrm{D}} = i\bar{\psi}\gamma^{\mu}\psi\,\mathrm{D}_{\mu}\delta U = \mathrm{D}_{\mu}\left(i\bar{\psi}\gamma^{\mu}\psi\,\delta U\right) - \mathrm{D}_{\mu}\left(i\bar{\psi}\gamma^{\mu}\psi\right)\delta U$$

from which we expect to infer a current density $(j_{\mu})_{\mathrm{D}}$ through a classical action variation

$$\delta\mathbf{S}_{\mathrm{D}} = -\int \mathrm{D}_{\mu}(j^{\mu})_{\mathrm{D}}\,\delta U\boldsymbol{\eta}\,, \tag{23}$$

where we have set

$$(j^{\mu})_{\mathrm{D}} = i\bar{\psi}\gamma^{\mu}\psi\,. \tag{24}$$

If ψ is a solution of the Dirac field equations (19), $\delta\mathbf{S}_{\mathrm{D}}$ vanishes for any δU, so

$$\mathrm{D}_{\mu}(j^{\mu})_{\mathrm{D}} = 0\,. \tag{25}$$

Thus we have defined the *conserved Dirac current vector density* $(j^{\mu})_{\mathrm{D}}$ which is a *real vector*. To prove this, we write $(j^{\mu})_{\mathrm{D}}$ with the aid of (12)

$$(j^{\mu})_{\mathrm{D}} = i\,\mathbf{t}\widetilde{\psi}\beta\gamma^{\mu}\psi\,.$$

Applying the usual adjoint operation $(j^{\mu})^{*}_{D} = -i\,\mathbf{t}\widetilde{\psi}\widetilde{\gamma}^{\mu}\beta\psi$ and taking into account $\widetilde{\gamma}^{\mu}\beta = -\beta\gamma^{\mu}$, we eventually find

$$(j^{\mu})^{*}_{\mathrm{D}} = i\,\mathbf{t}\widetilde{\psi}\gamma^{\mu}\psi = i\bar{\psi}\gamma^{\mu}\psi = (j^{\mu})_{\mathrm{D}} \tag{26}$$

which concludes the demonstration.

§4. The Dirac-Zelmanov equation

§4.1. The unique physically observable spinor. We shall now suggest a way to express the 1-contravariant spinor through the characteristics of Zelmanov's *formalism of General Relativity and Riemannian geometry* (i.e. the theory of chronometric invariants).

Consider the monad world-vector

$$b^{\mu} = \frac{dx^{\mu}}{ds}\,. \tag{27}$$

According to Zelmanov's theorem, for any vector Q^{μ}, two quantities are physically observable:

$$b^{\mu}Q_{\mu} = \frac{Q_0}{\sqrt{g_{00}}} \tag{28}$$

and
$$h^i_\mu Q^\mu = Q^i. \tag{29}$$

At first glance, all one would have to do, is simply replace Q^μ with j^μ. However, in this case, we readily see that Q_0 (or j_0) must differ from Q^0 (or j^0) which seems to contradict the matrix definition for γ^0 (7). Therefore, we will tackle the problem in a different manner.

With the aid of the standard gamma matrices (7), the components of j^μ are easily derived:

$$\left.\begin{aligned} j^0 &= \mathbf{t}\left(\psi^1\psi^{1*} + \psi^2\psi^{2*} + \psi^3\psi^{3*} + \psi^4\psi^{4*}\right) \\ j^1 &= -\mathbf{t}\left(\psi^2\psi^{1*} + \psi^1\psi^{2*} - \psi^4\psi^{3*} - \psi^3\psi^{4*}\right) \\ j^2 &= -i\,\mathbf{t}\left(\psi^2\psi^{1*} - \psi^1\psi^{2*} - \psi^4\psi^{3*} + \psi^3\psi^{4*}\right) \\ j^3 &= -\mathbf{t}\left(\psi^1\psi^{1*} - \psi^2\psi^{2*} - \psi^3\psi^{3*} + \psi^4\psi^{4*}\right) \end{aligned}\right\}. \tag{30}$$

The vector j^μ retains all the space-time properties as j^0 is shown to lie within the (future) half-light cone for $\mathbf{t} = +1$. Analogously, the vector $\boldsymbol{j}$ can be isotropic in which case $j^\mu j_\mu$ must be zero.

Like in the Riemannian picture, we make explicit $j^\mu j_\mu$ as follows:

$$j^\mu j_\mu = (j^0)^2 - (j^i)^2 = (j^0)^2 - j^i j_i\,. \tag{31}$$

Then, we remark that (31) is formally similar to Zelmanov's expression for an arbitrary vector A^μ

$$A^\mu A_\mu = a^2 - a^i a_i = a^2 - h_{ik}\,a^i a^k \tag{32}$$

setting $j^\mu = A^\mu$, we then have

$$a = \frac{j_0}{\sqrt{g_{00}}}\,, \tag{33}$$

$$a^i = j^i, \tag{34}$$

with

$$j^0 = \frac{a + \frac{v_i a^i}{c}}{1 - \frac{\mathrm{w}}{c^2}}\,, \qquad j_i = -a_i - \frac{v_i\,a}{c}\,, \tag{35}$$

where $v_i = -\frac{c g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of space rotation, while $\mathrm{w} = c^2(1-\sqrt{g_{00}})$ is the gravitational potential.

Thus, we see that the only combination of the observable spinor components are (with $\mathbf{t} = +1$, see above)

$$\mathbf{t}\left(\psi^1\psi^{1*} + \psi^2\psi^{2*} + \psi^3\psi^{3*} + \psi^4\psi^{4*}\right) = \frac{a + \frac{v_i a^i}{c}}{1 - \frac{\mathrm{w}}{c^2}}\,. \tag{36}$$

We have now reached our goal by linking the first spinor combination of (30) with relations (34) and (36), i.e. we have found the unique physically observable spinors. This mathematical approach also enables us to note that j_0 and j^0 have distinct expressions within the framework of Zelmanov's theory.

§4.2. The Zelmanov spinor connection. Consider now the Riemannian connection coefficients $\Gamma^{\mu}_{\nu\alpha}$ which are just the Christoffel symbols (i.e. Levi-Civita connection) with respect to any coordinate basis.

We define the *Zelmanov spinor connection* as

$$\left(N^{A}_{B\alpha}\right)_{\mathrm{Zel}} = -\frac{1}{4}\,(\Gamma^{\mu}_{\nu\alpha})_{\mathrm{Zel}}\,\gamma^{A}_{\mu C}\,\gamma^{\nu C}_{B}\,. \tag{37}$$

The components of the $(\Gamma^{\mu}_{\nu\alpha})_{\mathrm{Zel}}$ have been deduced by Zelmanov as the unique physically observable quantities

$$\Gamma^{0}_{00} = -\frac{1}{c^3}\left[\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right)v_k F^k\right],$$

$$\Gamma^{k}_{00} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F^k,$$

$$\Gamma^{0}_{0i} = \frac{1}{c^2}\left[-\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial x^i} + v_k\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}\,v_i F^k\right)\right],$$

$$\Gamma^{k}_{0i} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D^k_i + A^{\cdot k}_{i\cdot} + \frac{1}{c^2}\,v_i F^k\right),$$

$$\begin{aligned}\Gamma^{0}_{ij} = -\frac{1}{c\left(1-\frac{\mathrm{w}}{c^2}\right)}\Bigg\{&-D_{ij} + \frac{1}{c^2}\,v_n \times \\ &\times\left[v_j\left(D^n_i + A^{\cdot n}_{i\cdot}\right) + v_i\left(D^n_j + A^{\cdot n}_{j\cdot}\right) + \frac{1}{c^2}\,v_i v_j F^n\right] + \\ &+\frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right) - \Delta^n_{ij}\,v_n\Bigg\},\end{aligned}$$

$$\Gamma^{k}_{ij} = \Delta^k_{ij} - \frac{1}{c^2}\left[v_i\left(D^k_j + A^{\cdot k}_{j\cdot}\right) + v_j\left(D^k_i + A^{\cdot k}_{i\cdot}\right) + \frac{1}{c^2}\,v_i v_j F^k\right].$$

where

$$\Delta^i_{jk} = \frac{1}{2}h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right)$$

are the chronometrically invariant Christoffel symbols, for which the fundamental differential operator is

$$\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}.$$

Associated with the global non-holonomic vector v_i, Zelmanov's angular momentum tensor A_{ik} — ultimately characterizing the space as non-holonomic and non-isotropic — is given by

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}(F_i v_k - F_k v_i)\,.$$

Here

$$D_{ik} = \frac{1}{2\sqrt{g_{00}}}\frac{\partial h_{ik}}{\partial t}$$

is the space deformation tensor and

$$F_i = \frac{1}{1-\frac{\mathrm{w}}{c^2}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right)$$

is the gravitational inertial force vector.

§4.3. The massive Dirac field interacting with an electromagnetic field. Let us consider here the Lagrangian for a charged Dirac massive field coupled with a potential A_μ

$$\mathcal{L}(\psi,\, A_\mu) = \mathcal{L}(\psi) + \mathcal{L}(A_\mu) - e\,\psi A_\mu (j^\mu)_{\mathrm{D}}$$

the coupling constant e is taken as a negative charge (i.e. the electron).

Taking into acccount the expression of the Dirac current density

$$(j^\mu)_{\mathrm{D}} = i\,(\bar{\psi}\gamma^\mu\psi) \tag{38}$$

we shall evaluate the variation of the Lagrangian $\mathcal{L}(\psi, A_\mu)$.

After a simple but lengthy calculation, we obtain (omitting $_{\mathrm{D}}$ in j)

$$\begin{aligned} \delta\mathcal{L}(\psi, A_\mu) &= \left[\delta\,\bar{\psi}\left(\gamma^\mu(\mathrm{D}_\mu - i\,eA_\mu)\,\bar{\psi} - m_0 c\psi\right)\right] + \\ &+ \left[\left(-(\mathrm{D}_\mu + i\,eA_\mu)\bar{\psi}\gamma^\mu - m_0 c\psi\right)\delta\psi\right] + \\ &+ \left[\,-\mathrm{D}^\nu(\mathrm{D}_\nu A_\mu - \mathrm{D}_\mu A_\nu) - e\,j_\mu\right]\delta A^\mu + \text{divergence term.} \end{aligned} \tag{39}$$

Extremalizing the action defined from $\mathcal{L}(\psi, A_\mu)$, we get two (massive) field equations

$$\gamma^\mu(\mathrm{D}_\mu - i\,eA_\mu)\,\psi = m_0 c\psi\,, \tag{40}$$

$$-(\mathrm{D}_\mu + i\,eA_\mu)\,\bar{\psi}\gamma^\mu = m_0 c\bar{\psi}\,, \tag{41}$$

and

$$\delta\, d\boldsymbol{A} = e\,\boldsymbol{j}\,. \tag{42}$$

Under the conjugate operation, the following transformation

$$\psi \longrightarrow \psi_{(c)}$$

takes a place, and the first equation (40) becomes

$$\gamma^{\mu}\left(\mathrm{D}_{\mu} + i\,eA_{\mu}\right)\psi_{(c)} = m_0 c\,\psi_{(c)} \tag{43}$$

and it is interpreted as the *positron equation* which accounts for the *anti-electron* or *positron* with a *positive* charge, when $\psi_{(c)}$ is substituted into ψ. Thus, in equation (43), the rest mass m_0 represents the positron.

Within the Zelmanov picture, the Dirac equation (40) will be uniquely written as

$$\gamma^{\mu}\left[(\mathrm{D}_{\mu})_{\mathrm{Zel}} - i\,e\,(A_{\mu})_{\mathrm{Zel}}\right]\psi' = m_0 c\,\psi'\,, \tag{44}$$

where ψ' is the modified spinor according to (36) and $(\mathrm{D}_{\mu})_{\mathrm{Zel}}$ is the spinor derivative constructed from the components of (37).

As regards the four-potential $(A_{\mu})_{\mathrm{Zel}}$, we have the following components

$$A_0 = b^{\mu} A_{\mu} \sqrt{g_{00}}\ . \tag{45}$$

This scalar potential is a chronometrically invariant quantity, with the associated vector components:

$$A_i = -\,h_{ik}A^k - \frac{A_0\, v_i}{c\sqrt{g_{00}}}\,. \tag{46}$$

Concluding remarks. There is no ambiguity neither any loss of generality regarding the special choice of gamma matrices (7), as long as they verify the fundamental relation (6). Therefore, assuming that the inferred Dirac current is a space-time vector is here legitimate all the more as this vector can be isotropic.

Based on this weak constraint, it has thus been possible to express the contravariant 1-spinor ψ (or its combination) in terms of the gravitational potential w and the linear velocity of space rotation v_i.

The Dirac-Zelmanov equation for a massive field completes the scope of the theory by implicitly displaying the *non-holonomity* and *non-isotropy tensor* A_{ik} and the *tensor of deformation of space* D_{ik} through the Zelmanov spinor connection.

In equation (44), the electron and its rest mass m_0 are constant and are independent of the observer, so even when the contravariant 1-spinor

ψ is modified when viewed in the observer's physical frame, the positron equation (43) should also be true in Zelmanov's theory.

But above all, the most salient feature of the present theory certainly lies in that the Zelmanov theory entirely *confirms* the way Lichnérowicz approached the spinor analysis in General Relativity.

It is indeed remarkable to note that the observable spinor formulation requires the fundamental relation (6) to be maintained.

Any other current attempts to write down (6) as

$$\gamma_a(x)\,\gamma_b(x) + \gamma_b(x)\,\gamma_a(x) = -2\,g_{ab}\,\mathbf{I},$$

where $\gamma_a(x) = a^{\mu}_{a}(x)\gamma_{\mu}$ are the mere local generalization of the gamma matrices (see for instance [11, p. 25] or [12, p. 515]), can be definitely ruled out.

It also means that the adopted metric form $ds^2 = \eta_{\mu\nu}\,\theta^{\mu}\theta^{\nu}$ clearly appears to be the right choice for describing the spinor in General Relativity.

This is certainly the *essential* result of our short study, as it dramatically shows that the chronometric (physically observable) properties of Zelmanov are equivalent to a pure mathematical analysis (Lichnérowicz) in perfect harmony with quantum theory (Dirac) and resulting experimental data (i.e. existence of the positron).

Appendix. In mathematics, a vector cannot in general be considered as an arrow connecting two points on the manifold M [14]. A tangent vector $\boldsymbol{V}$ along a curve $\gamma(t)$ at p, is considered as an operator (linear functional) which assigns to each differential function f on M, a real number $\boldsymbol{V}(f)$.

This operator satisfies the axioms:

Axiom I: $\boldsymbol{V}(f+h) = \boldsymbol{V}(f) + \boldsymbol{V}(h)$,

Axiom II: $\boldsymbol{V}(fh) = h\boldsymbol{V}(f) + f\boldsymbol{V}(h)$,

Axiom III: $\boldsymbol{V}(cf) = c\boldsymbol{V}(f)$, where $c = constant$.

One shows that such a tangent vector can be expressed by

$$\boldsymbol{V} = V^a \frac{\partial}{\partial x^a}\,,$$

where the real coefficients V^a are the components of $\boldsymbol{V}$ at p, with respect to the *local* coordinates $(x^1, \ldots, x^4)$ in the neighbourhood of p. Accordingly, the directional derivatives along the coordinates lines at p form a basis of the four-dimensional vector space whose elements are the tangent vectors at p, i.e. the tangent space T_p.

The basis $(\frac{\partial}{\partial x^a})$ is called a *coordinate basis*. On the contrary, a general basis e_ν is formed by four linearly independent vectors e_ν; any vector $\boldsymbol{V} \in \mathrm{T}_p$ is a *linear combination* of these basis vectors:

$$\boldsymbol{V} = V^\alpha e_\alpha \,.$$

By definition, a *1-form* (*Pfaffian form*) ζ maps a vector $\boldsymbol{V}$ into a real number, with the *contraction* denoted by the symbol $<\zeta, \boldsymbol{V}>$, and this mapping is *linear*.

The four linearly independent 1-forms θ^μ which are uniquely determined by

$$<\theta^\mu, e_\nu> = \delta^\mu_\nu$$

form a general basis of the dual space T^*_p to the tangent space T_p. This basis is said to be dual to the basis e of T_p. The bases e_ν, θ^μ are linear combinations of coordinates bases:

$$e_\nu = b^a_\nu \left(\frac{\partial}{\partial x^a}\right), \qquad \theta^\mu = a^\mu_a dx^a.$$

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