Gravitational Fields Exterior to Homogeneous Spheroidal Masses

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Abstract: General relativistic mechanics in gravitational fields exterior to homogeneous spheroidal masses is developed using our new approach. Einstein's field equations in the gravitational field exterior to a static homogeneous prolate spheroid are derived and a solution for the first field equations constructed. Our derived field equations exterior to the mass distribution have only one unknown function determined by the mass or pressure distribution. The obtained solutions yield the unknown function as generalizations of Newton's gravitational scalar potential. Remarkably, our solution puts Einstein's geometrical theory of gravity on same footing with Newton's dynamical theory; with the dependence of the field on one and only one unknown function comparable to Newton's gravitational scalar potential. The consequences of the homogeneous spheroidal gravitational field on the motion of test particles have been theoretically investigated. The effect of the oblate nature of the Sun and planets on some gravitational phenomena has been examined. These are gravitational time dilation, gravitational length contraction and gravitational spectral shift of light. Our obtained theoretical value for the Pound-Rebka experiment on gravitational spectra shift (2.578×10^{-15}) agrees satisfactorily with the experimental value of 2.45×10^{-15}. Expressions for the conservation of energy and angular momentum are obtained. Planetary equations of motion and equations of motion of photons in the vicinity of spheroids are derived; having additional spheroidal terms not found in Schwarzschild's space-time.

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Chapter 1. Introduction

§1.1. Background of the problem

§1.1.1 The nature of gravitation

General Relativity is the geometrical theory of gravitation published by Albert Einstein in 1915/1916. It unifies Special Relativity and Sir Isaac Newton's law of universal gravitation with the insight that gravitation is not due to a force but rather a manifestation of curved space and time, with the curvature being produced by the mass-energy and momentum content of the space-time. General Relativity is the most widely accepted theory of gravitation.

After his theory of Special Relativity which elegantly describes mechanics in electromagnetic and empty spaces, Einstein expected gravitation to have the same nature as electromagnetism and hence fit into Special Relativity. So Einstein sought a "Maxwellian" type of laws for the gravitational field. That effort by Einstein failed [1]. Einstein concluded that gravitation is of an entirely different nature from electromagnetism which is a dynamical phenomenon. Consequently, he used geometrical quantities (tensors) for the description of gravitation instead of the dynamical quantities such as force and potential. Secondly, Einstein realized that Newton's laws of gravitation satisfied Galileo's principle of relativity according to which the laws of physics take the same form in all inertial reference frames. Consequently, Einstein also introduced his principle of General Relativity which asserts that "The laws of physics take the same form in all reference frames" Thus, Einstein constructed his theory of gravitation founded on his principle of General Relativity using tensors [1].

§1.1.2. The space-time of General Relativity

In Special Relativity, space-time has four dimensions ($\mu = 0, 1, 2, 3$) and there always exist a global coordinate system in which the world-line element or proper time takes the form

$$c^2 d\tau^2 = \eta_{\mu\sigma}\, dx^\mu dx^\sigma , \tag{1.1}$$

where $\eta_{\mu\sigma}$ is a special relativistic metric tensor given by $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$ ($\eta_{\mu\sigma} = 0$, $\mu \neq \sigma$). Such a coordinate system is said to be Cartesian. In a non-Cartesian coordinate system such as a spherical or spheroidal coordinates, the world-line element of space-time may be written as [2],

$$c^2 d\tau^2 = g_{\mu\sigma}\, dx^\mu dx^\sigma , \tag{1.2}$$

where $g_{\mu\sigma}$ is the corresponding metric tensor which is generally different from the Cartesian metric tensor $\eta_{\mu\sigma}$. In practical calculations, the metric is most often written in coordinates in which it takes the following form

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu. \tag{1.3}$$

According to the philosophy of General Relativity (GR), the effect of gravitation is contained in the metric tensor field $g_{\mu\sigma}$. Thus, in Einstein's theory of gravity, the gravitational field is promoted to a space-time metric $g_{\mu\sigma}$.

Schwarzschild in 1916 constructed the first exact solution of Einstein's gravitational field equations. Schwarzschild's solution is one of the physically interpretable solutions of Einstein's field equations [3]. Schwarzschild metric tensor field is that due to a static spherically symmetric body situated in empty space such as the Sun or a star. Schwarzschild metric has been the basis of theoretical investigations of gravitational phenomena in Einstein's theory of gravitation. This is in spite of the fact that the Sun and most planetary bodies in the Solar System are not perfectly spherical but oblate spheroidal in shape [4].

§1.2. Statement of the problem

From the inception of Newton's dynamical theory of gravitation in the 17th century, the planets and Sun have been treated as perfectly spherical bodies. For example in the motion of terrestrial penduli, projectiles and satellites, the Earth is regarded as perfectly spherical in geometry. Similarly in the motions of the planets, comets and asteroids in the Solar System, the Sun is regarded as perfectly spherical in geometry and also, in Einstein's geometrical theory of gravitation (General Relativity). The motions of the planets and photons in the Solar System are treated under the assumption that the Sun is a perfect sphere. It has however, been realized experimentally that the Sun and planets in the Solar System are more precisely oblate spheroidal in geometry [4] (see Table 1 below).

Obviously, the oblate spheroidal geometries of these bodies (Table 1) has corresponding effects on their gravitational fields and hence the motions of test particles in these fields. Towards the investigation of these effects in Newton's theory of gravitation; the gravitational scalar potential due to an oblate spheroidal body and Newton's equations of motion in the gravitational field of an oblate mass have been derived [4].

The prolate spheroid is the shape of some moons in the Solar System. Examples are Mimas, Enceladus, and Tethys (moons of Saturn) and

Body	Oblateness
Sun	9×10^{-6}
Mercury	0
Venus	0
Earth	0.0034
Mars	0.006
Jupiter	0.065
Saturn	0.108
Uranus	0.03
Neptune	0.026

Table 1: Oblateness of bodies in the Solar System.

Miranda (moon of Uranus). The prolate spheroidal geometry is also used to describe the shape of some nebulae (a nebula is a region or cloud of interstellar dust and gas appearing variously as a hazy bright or dark patch) such as the Crab Nebula [5]. Also, the existence of rotating prolate spheroidal galaxies has been known for decades, yet, a theoretical model based on Newton's or Einstein's gravitational theories remains elusive [6].

The metric tensor for a gravitational field is the fundamental starting point in the studies of gravitational fields in Einstein's geometrical theory. With the metric tensor, Einstein's field equations can be derived and solved. There is no general method yet of finding rigorous solutions of Einstein's field equations [1]. In order to study general relativistic mechanics (Einstein's theory of gravitation) in oblate spheroidal gravitational fields, Howusu and Uduh [7] sought the covariant metric tensor exterior or interior to a massive oblate spheroidal body in oblate spheroidal coordinates as;

$$g_{00} = e^{-F}, \tag{1.4}$$

$$g_{11} = -\,e^{-G}, \tag{1.5}$$

$$g_{22} = -\,e^{-H}, \tag{1.6}$$

$$g_{33} = -\,a^2\left(1-\eta^2\right)\left(1+\zeta^2\right), \tag{1.7}$$

$$g_{\mu\nu} = 0\,, \tag{1.8}$$

where F, G and H are functions of η and ζ only and a is a constant parameter. With this metric, they constructed gravitational field equations exterior or interior to a massive oblate spheroidal body. The field equations they obtained are non linear second order differential equations and have three unknown functions. The major setback of the use of this metric, equations is that the introduction of three unknown functions, F, G and H makes the field equations obtained very complex and this compounds with the non linearity of the field equations to make them almost practically unsolvable and physically uninteresting.

Howusu [8] in an attempt to address the loop holes, difficulties and shortcomings in their previous approach, realized that a general and standard metric tensor exterior to all distributions of mass or pressure within regions of all regular geometries can be obtained by extending the Schwarzschild's metric to the particular regular geometry. The most interesting and important fact about this new method is that the generalized metric tensor obtained is not an exponential function and has only one unknown function. It is also instructive to note that the unknown function in this case can be satisfactorily approximated to the Newtonian gravitational scalar potential exterior to the astrophysical body under consideration and hence makes physical interpretations simpler. This new approach is thus computationally less cumbersome and physically more applicable in principle than the previous approach.

This work examines the effect of the oblate spheroidal nature of the Sun and planets on some gravitational phenomena using this new approach. The motion of planets and photons in the Solar System are also investigated. These effects include gravitational spectral shift of light, gravitational length contraction and gravitational time dilation. We equally construct the generalized Lagrangian for this field and use it to study orbits in homogeneous oblate spheroidal space-time [9–11]. In this research work, we also start the study of static homogeneous prolate spheroidal gravitational fields using this new approach. Einstein‘s gravitational field equations exterior to static homogenous prolate spheroids are derived. Solutions to the derived field equations are also constructed and the consequences of the field on the motion of test particles are also investigated [12, 13].

§1.3. Objectives of the study

- Derivation of Einstein's gravitational field equations exterior to static homogeneous (time independent) prolate spheroidal distributions of mass as an extension of Schwarzschild's metric (that is using the new approach);
- Solutions to field equations derived and consequences to the motion of test particles;

- Derivation of the planetary equation of motion and the equation for the deflection of light in the gravitational field exterior to homogenous (time independent) spheroids (prolate and oblate);
- Investigation of the effect of the oblate spheroidal nature of the Sun and planets on some gravitational phenomena (gravitational length contraction, gravitational time dilation and gravitational spectral shift of light) using the new approach.

§1.4. Scope of the study

The philosophy of General Relativity describes gravitation as a geometrical phenomenon with the effect of gravitation contained in the covariant metric tensor for a gravitational field [14]. The metric tensor exterior to all possible distributions of mass within oblate spheroidal and prolate spheroidal geometries given by Howusu [8] is made explicit and used to study these gravitational fields. Our knowledge of orthogonal curvilinear coordinates, tensor analysis, Schwarzschild gravitational field and general relativistic mechanics is used to achieve the objectives.

Basically, we concentrate on gravitational sources with time independent and axially-symmetric distributions of mass within spheroids, characterized by at most two typical integrals of geodesic motion, namely, energy and angular momentum. From an astrophysical point of view, such an assumption, although not necessary, could, however, prove useful, because it is equivalent to the assumption that the gravitational source is changing slowly in time so that partial time derivatives are negligible compared to the spatial ones. We stress that the mass source considered is not the most arbitrary one from a theoretical point of view, but on the other hand, many astrophysically interesting systems are usually assumed to be time independent (or static from another point of view) and axially symmetric continuous sources [15].

§1.5. Significance of the study

Gravity is the least understood of all the fundamental forces in nature; but mass and space, which are governed by gravity, are the building blocks and fabric of our universe. General Relativity is the most fundamental theorem of physics about the nature of gravity. If we better understand the nature of mass and space, we may be able to do things previously undreamed of. So far studies of General Relativity have yielded atomic clocks, guidance systems for spacecrafts and the Global Positioning Systems (GPS). We cannot foresee all that can come from a better understanding of space-time and mass-energy, but a theorem about these fundamental subjects must be thoroughly examined if we are to use it to our advantage [16]. This research work is a step in this direction.

This research work substantially extends Einstein's theory of gravitation (General Relativity) from the well known Schwarzschild space-time to the experimentally more precise oblate and prolate spheroidal space-times in the universe. Thus, the theoretical analysis of the motion of particles of non-zero rest masses, gravitational length contraction, gravitational time dilation and gravitational spectral shift is extended from the gravitational field exterior to a spherical mass to the gravitational field exterior to spheroidal masses. Our approach in this research work unlike in earlier attempts makes it possible for us to obtain physically interpretable theoretical values for the above listed gravitational phenomena in approximate gravitational fields exterior to bodies in the Solar System. Our newly obtained expression for gravitational time dilation can now be incorporated into the contribution of gravitation in the design of Global Positioning System (GPS). It is hoped that when this is done, the precision rate of GPS will be greatly improved. This work also opens the door for the theoretical investigation of the contributions of the oblateness of the Sun and planets on other gravitational phenomena such as geodetic deviation, radar sounding and anomalous orbital precession, using this new approach. An insight is also provided for the theoretical investigation of the contributions of the oblateness and prolateness of some astronomical bodies on gravitational phenomena. It is thus eminent that this work will serve as an eye opener for the verification of small departures of theory from reality in astronomy in the near future.

Chapter 2. Methodology

§2.1. General relativistic mechanics in Schwarzschild's field

§2.1.1. Einstein's gravitational field equations

It is well known that Einstein's gravitational field equations are tensorially given as [1]

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}\, T_{\mu\nu}\,, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor constructed from the metric tensor $g_{\mu\nu}$ of the space-time; c is the speed of light in vacuum, G is the universal gravitational constant and $T_{\mu\nu}$ is the stress tensor which is the source of the gravitational metric field. There are actually ten independent scalar equations in (2.1) because of the symmetry of the tensors involved in the field equations. These equations are actually second order partial differential equations and are generally non-linear.

§2.1.2. Schwarzschild's metric

If one considers a spherical body of radius R_0 and total rest mass M distributed uniformly with density ρ_0, then the general relativistic field equations in its exterior region are given tensorially as [1]

$$G_{\mu\nu} = 0\,. \tag{2.2}$$

Thus, Einstein's equations (2.2) give ten different differential equations with zero elements on the right-hand-side. Schwarzschild in 1916 constructed the first exact solution of Einstein's gravitational field equations. It was the metric due to a static spherically symmetric body situated in empty space such as the Sun or a star [1]. The result is as follows

$$g_{00} = 1 + \frac{2f(r)}{c^2}\,, \tag{2.3}$$

$$g_{11} = -\left(1 + \frac{2f(r)}{c^2}\right)^{-1}, \tag{2.4}$$

$$g_{22} = -r^2, \tag{2.5}$$

$$g_{33} = -r^2 \sin^2\theta\,, \tag{2.6}$$

where $f(r)$ is an arbitrary function determined by the distribution. It is a function of the radial coordinate r only; since the distribution and hence its exterior gravitational field posses spherical symmetry. From the condition that these metric components should reduce to the field of a point mass located at the origin and contain Newton's equations of motion in the gravitational field of the spherical body, it follows that $f(r)$ is the Newtonian gravitational scalar potential in the exterior region of the body [17].

§2.1.3. Schwarzschild's singularity

The world-line element in Schwarzschild field is given by [8]

$$c^2 d\tau^2 = c^2\left(1 + \frac{2f(r)}{c^2}\right)dt^2 - \left(1 + \frac{2f(r)}{c^2}\right)^{-1} dr^2 - \\ - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2. \tag{2.7}$$

In this field

$$f(r) = -\frac{GM}{r}\,, \qquad r > R_0\,. \tag{2.8}$$

It has been known since 1916 that it is possible for a spherical body to have a point outside it at which the Schwarzschild metric has a singularity. This singularity is denoted by r_s and is called the Schwarzschild singularity. It is given by the condition

$$1 - \frac{2GM}{c^2 r_s} = 0\,,$$

thus,

$$r_s = \frac{2GM}{c^2}\,. \tag{2.9}$$

For the Earth, $r_s = 0.89$ cm. This radius lies in the interior of the Earth where the metric is precisely the interior metric and hence the exterior metric is not applicable. For most physical bodies in the universe, the Schwarzschild radius is much smaller than the radius of their surface. Hence for most bodies, there does not exist a Schwarzschild singularity. It is however, speculated that there exist some bodies in the universe with the Schwarzschild radius in the exterior region. Such bodies are called black holes.

§2.1.4. Gravitational length contraction in Schwarzschild field

In Schwarzschild field, the space part of the metric is given by

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{2.10}$$

Thus, in the neighborhood of a massive body, two points of the same angle θ and ϕ now have a separation which is different from the corresponding separation in empty space. That is

$$ds = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr \,\cong\, \left(1 + \frac{GM}{c^2 r} + \cdots\right) dr\,. \tag{2.11}$$

This equation implies that $ds > dr$. In other words r is no longer the measure of radial distances. Also, it follows that the length of physical bodies is not conserved in a gravitational field. That is, length is contracted in a gravitational field. This is the phenomenon of length contraction. It is highly speculated that not only are material objects (such as meter rules) contracted by gravitational fields but also space itself is contracted by gravitational fields [2].

§2.1.5. Gravitational time dilation in the spherical field

Consider, a clock at rest at a fixed point in Schwarzschild gravitational field around a spherical body, then $dr = d\theta = d\phi = 0$ and hence Schwarzschild's world line element, reduces to

$$dt = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} d\tau \cong \left(1 + \frac{GM}{c^2 r} + \cdots\right) d\tau \,. \tag{2.12}$$

It can be deduced that $dt > d\tau$ and therefore, the coordinate time of a clock in the gravitational field is dilated relative to the proper time.

§2.1.6. Motion of particles of non-zero rest masses in Schwarzschild field

A test mass is one which is so small that the gravitational field produced by it is so negligible that it does not have any effect on the space metric. A test mass is a continuous body, which is approximated by its geometrical centre; it has nothing in common with a point mass whose density should obviously be infinite [1].

The general relativistic equation of motion for particles of non-zero rest masses in a gravitational field are given by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^2}{d\tau} = 0 \,, \tag{2.13}$$

where $\Gamma^\mu_{\nu\lambda}$ are the coefficients of affine connection for the gravitational field. For Schwarzschild field, the equations of motion are

$$\ddot{t} + \frac{k}{c^2 r^2 \left(1 - \frac{2k}{c^2 r}\right)} \, \dot{t}\dot{r} = 0 \,, \tag{2.14}$$

$$\ddot{\theta} + \frac{2}{r} \, \dot{r}\dot{\theta} - \sin\theta \, \cos\vartheta \, \dot{\phi}^2 = 0 \,, \tag{2.15}$$

$$\ddot{\phi} + \frac{2}{r} \, \dot{r}\dot{\phi} + 2 \cot\theta \, \dot{\theta} \dot{\phi}^2 = 0 \,, \tag{2.16}$$

$$\ddot{r} + \frac{1}{2} \, c^2 f^1 \left(1 + f\right) \dot{t}^2 - \frac{1}{2} \, f^1 \left(1 + f\right)^{-1} \dot{r}^2 - \\ - r \left(1 + f\right) \dot{\theta}^2 - r \left(1 + f\right) \sin^2\theta \, \dot{\phi}^2 = 0 \,, \tag{2.17}$$

where the dot denotes differentiation with respect to proper time, $k \equiv GM$, $f \equiv -\frac{2k}{c^2 r}$ and $f^1 \equiv \frac{df}{dr}$ [1].

§2.2 General relativistic mechanics in static homogeneous spheroidal fields

§2.2.1. Oblate and prolate spheroidal coordinate systems

The oblate spheroidal coordinates are related to the Cartesian coordinates by

$$\left.\begin{aligned} x &= a\cosh u\cos v\cos\phi \\ y &= a\cosh u\cos v\sin\phi \\ z &= a\sinh u\sin v \end{aligned}\right\}, \qquad (2.18)$$

where $u \geqslant 0$, $0 \leqslant v \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$.

It is convenient to use the following transformations to eliminate the hyperbolic functions and ease computation with this coordinate system; $\xi = \sinh u$, $\eta = \sin v$ and thus, the relation between Cartesian and oblate spheroidal coordinate systems can be written as [18]

$$\left.\begin{aligned} x &= a\left(1-\eta^2\right)^{1/2}\left(1+\xi^2\right)^{1/2}\cos\phi \\ y &= a\left(1-\eta^2\right)^{1/2}\left(1+\xi^2\right)^{1/2}\sin\phi \\ z &= a\eta\xi \end{aligned}\right\}, \qquad (2.19)$$

where $0 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant 1$, $0 \leqslant \phi \leqslant 2\pi$ and a is a constant parameter.

For a prolate spheroid unlike an oblate spheroid, the polar diameter is longer than the equatorial diameter. The derivation of the prolate spheroidal coordinate system is quite similar to the above derivation of the oblate spheroidal coordinate system. The relation between the Cartesian and prolate spheroidal coordinate systems is [18]

$$\left.\begin{aligned} x &= a\left(1-\eta^2\right)^{1/2}\left(1+\xi^2\right)^{1/2}\cos\phi \\ y &= a\left(1-\eta^2\right)^{1/2}\left(1+\xi^2\right)^{1/2}\sin\phi \\ z &= a\eta\xi \end{aligned}\right\}, \qquad (2.20)$$

where $0 \leqslant \xi < \infty$, $-1 \leqslant \eta \leqslant 1$ and $0 \leqslant \phi \leqslant 2\pi$.

§2.2.2. Metric tensor exterior to an oblate spheroid and a prolate spheroid

The invariant world line element in the exterior region of a static spherical body is given generally according to [8], where $f(r,\theta,\phi)$ is a generalized arbitrary function determined by the distribution of mass or pressure and possess all the symmetries of the mass distribution. Thus, according to [8], the invariant world line element is

$$c^2 d\tau^2 = c^2 \left(1 + \frac{2f(r,\theta,\phi)}{c^2}\right) dt^2 - \\ - \left(1 + \frac{2f(r,\theta,\phi)}{c^2}\right)^{-1} dr^2 - r^2 \sin^2\theta \, d\phi^2. \qquad (2.21)$$

It is a well known fact of General Relativity that $f(r,\theta,\phi)$ is approximately equal to Newton's gravitational scalar potential in the space-time exterior to the mass or pressure distributions within spherical geometry.

Now, let the spherical body be transformed, by deformation, into an oblate spheroidal body in such a way that its density ρ_0 and total mass M remain the same and its surface parameter is given in oblate spheroidal coordinates as

$$\xi = \xi_0 = \text{constant}. \qquad (2.22)$$

Then, the general relativistic field equations exterior to an oblate spheroidal body are mathematically equivalent to those of the spherical body. This is because they are both tensorially the same. Hence, they are only related by the transformation from spherical to oblate spheroidal coordinates. Therefore, to get the corresponding invariant world line element in the exterior region of an oblate spheroidal mass one could do the following:

1) Replace $f(r,\theta,\phi)$ by the corresponding function $f(\eta,\xi,\phi)$ exterior to oblate spheroidal bodies. Thus, a sound and astrophysically satisfactory approximate expression for the function $f(\eta,\xi,\phi)$ is obtained by equating it to the gravitational scalar potential exterior to the distribution of mass within oblate spheroidal regions [8];
2) Transform coordinates from spherical to oblate spheroidal

$$(ct, r, \theta, \phi) \to (ct, \eta, \xi, \phi) \qquad (2.23)$$

on the right hand side of equation (2.7). The following components of the metric tensor in the region exterior to a homogeneous oblate spheroid in oblate spheroidal coordinates are obtained

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta,\xi)\right), \qquad (2.24)$$

$$g_{11} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2 \left(1 + \frac{2}{c^2} f(\eta,\xi)\right)^{-1} + \frac{\xi^2 \left(1+\xi^2\right)}{(1-\eta^2)}\right], \qquad (2.25)$$

$$g_{12} = g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2}\,f(\eta,\xi)\right)^{-1}\right], \tag{2.26}$$

$$g_{22} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}\,f(\eta,\xi)\right)^{-1}+\frac{\eta^2\left(1-\eta^2\right)}{\left(1+\xi^2\right)}\right], \tag{2.27}$$

$$g_{33} = -a^2\left(1+\xi^2\right)\left(1-\eta^2\right). \tag{2.28}$$

It may be of interest to note that this metric tensor field unlike the metric tensor field used by Howusu and Uduh [7] contains only one unknown function, $f(\eta,\xi)$ determined by the mass distribution and has no exponential components.

The covariant metric tensor obtained above for gravitational fields exterior to oblate spheroidal masses has two additional non-zero components g_{12} and g_{21} not found in Schwarzschild field and the metric used by Howusu and Uduh [7]. Thus, the extension from Schwarzschild field to homogeneous oblate spheroidal gravitational fields has produced two additional non zero tensor components and thus this metric tensor field is unique. This confirms the assertion that oblate spheroidal gravitational fields are more complex than spherical fields and hence general relativistic mechanics in this field is more involved. This partly accounts for the scanty research carried out on this gravitational field.

Similarly, it has been shown [8] that the covariant metric tensor exterior to static homogeneous prolate spheroidal distributions of mass is given as

$$g_{00} = \left(1+\frac{2}{c^2}\,f(\eta,\xi)\right), \tag{2.29}$$

$$g_{11} = -\frac{a^2\eta^2}{\eta^2+\xi^2-1}\left[\left(1+\frac{2}{c^2}\,f(\eta,\xi)\right)^{-1}+\frac{\xi^2\left(1-\xi^2\right)}{\eta^2\left(\eta^2-1\right)}\right], \tag{2.30}$$

$$g_{12} = g_{21} = -\frac{a^2\eta\xi}{\eta^2+\xi^2-1}\left[-1+\left(1+\frac{2}{c^2}\,f(\eta,\xi)\right)^{-1}\right], \tag{2.31}$$

$$g_{22} = -\frac{a^2\xi^2}{\eta^2+\xi^2-1}\left[\left(1+\frac{2}{c^2}\,f(\eta,\xi)\right)^{-1}+\frac{\eta^2\left(\eta^2-1\right)}{\xi^2\left(1-\xi^2\right)}\right], \tag{2.32}$$

$$g_{33} = -a^2\left(1-\xi^2\right)\left(\eta^2-1\right). \tag{2.33}$$

This metric tensor has the same number of non-zero components (six) as the metric exterior to an oblate spheroid. As has been noted in the case of oblate spheroids, $f(\eta, \xi)$ is an arbitrary function determined by the mass or pressure and hence it possesses all the symmetries of the latter, a priori. Herein, $f(\eta, \xi)$ can be conveniently approximated to be equal to Newton's gravitational scalar potential exterior to the mass distribution.

The metric tensors by virtue of their construction satisfy the first and second postulates of General Relativity. There are invariance of the line element; and Einstein's gravitational field equations [8].

§2.2.3. Gravitational field equations exterior to static homogeneous prolate spheroids

To obtain the contravariant metric tensor for the gravitational field exterior to a prolate spheroid, $g^{\mu\nu}$ we use the fact that $g^{\mu\nu}$ is the cofactor of $g_{\mu\nu}$ in g divided by g [18]. That is

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\mu\nu} \text{ in } g}{g}\,, \tag{2.34}$$

where

$$g = \det \begin{Vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{Vmatrix}. \tag{2.35}$$

The coefficients of affine connection $\Gamma^{\sigma}_{\mu\sigma}$ for any gravitational field are defined in terms of the covariant and contravariant metric tensor of space-time as [18]

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\, g^{\sigma v} \left(g_{\mu\nu,\,\lambda} + g_{v\lambda,\,\mu} - g_{\mu\lambda,\,v} \right), \tag{2.36}$$

where the comma denotes partial differentiation with respect to λ, μ and v. In this research work, we have constructed the 64 coefficients of affine connection for this gravitational field.

The curvature tensor or the Riemann-Christoffel tensor $R^{\delta}_{\alpha\beta\sigma}$ for this field is defined in terms of the coefficients of affine connection as

$$R^{\delta}_{\alpha\beta\sigma} = \Gamma^{\delta}_{\alpha\sigma,\,\chi} - \Gamma^{\delta}_{\alpha\beta,\,\sigma} + \Gamma^{\varepsilon}_{\alpha\sigma}\Gamma^{\delta}_{\varepsilon\beta} - \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\sigma}\,, \tag{2.37}$$

where the comma denotes partial differentiation with respect to β and σ. This research work has equally constructed the 256 components of this tensor for homogeneous prolate spheroidal gravitational fields. From the curvature tensor $R^{\delta}_{\alpha\beta\sigma}$ for this gravitational field, we have defined a second rank tensor $R_{\alpha\beta}$ (called the Ricci tensor) for the gravitational field exterior to the prolate spheroid as

$$R_{\alpha\beta} = R^{\delta}_{\alpha\beta\delta}\,. \tag{2.38}$$

The 16 components of this tensor for the static homogeneous prolate spheroids have been constructed. From the Ricci tensor for our gravitational field, we deduced a scalar R defined by

$$R = R^{\alpha}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta} \tag{2.39}$$

called the curvature scalar for homogeneous spheroidal fields.

It is well known that for a region exterior to any astrophysical body, the general relativistic field equations are given tensorially as

$$G_{\mu\nu} = 0\,, \tag{2.40}$$

where $G_{\mu\nu}$ is the Einstein tensor, given explicitly as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}\,, \tag{2.41}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the curvature scalar and $G_{\mu\nu}$ the covariant metric tensor for the field. The Einstein field equations for the gravitational field exterior to homogeneous prolate spheroids are then built up.

These are partial differential equations with only one unknown. We constructed the solution to the first field equation using our knowledge of partial differential equations and path integral methods.

§2.2.4. Motion of test particles exterior to static homogeneous prolate spheroidal masses

The general relativistic equation of motion and the coefficients of affine connection for our field are used to study the motion of particles of non-zero rest masses in this field. Einstein's geometrical equations of motion for test particles in the gravitational fields of prolate spheroidal astronomical bodies are derived. These equations of motion have only one unknown function. The solution of the first field equation is then used to study the effect of the gravitational field on the motion of test particles.

§2.2.5. Planetary motion and motion of photons in spheroidal gravitational fields

In General Relativity, the change in energy of a freely moving photon is given by the scalar equation of the isotropic geodesic equations, which manifest on the work produced on a photon being moved along a path [19]. Here, we use the generalized Lagrangian exterior to static homogenous oblate and prolate spheroids to study orbits in this gravitational field and obtain expressions for the conservation of total energy and angular momentum in this field. The planetary equation of motion and the equation for the deflection of light (photons) in the gravitational field exterior to homogeneous oblate and prolate spheroidal bodies are derived.

§2.2.6. Effects of oblateness of the Sun and planets on some gravitational phenomena

Gravitational time dilation. In this research work, we show that our theoretical extension of Schwarzschild's gravitational field to oblate spheroidal fields conform satisfactorily to the above proven experimental and astrophysical facts. We consider a clock at rest in this gravitational field such that $d\xi = d\eta = d\phi = 0$. The world line element for the gravitational field exterior to an oblate spheroidal mass is then used to give a new expression for time dilation. We then use our new expression to calculate the dilated coordinate time as a function of proper time along the equator and pole of various bodies in the Solar System in approximate homogeneous gravitational fields. This has not been done in previous theoretical approaches to the subject.

Gravitational length contraction. In this research work, the space part of the world line element in the gravitational field exterior to an oblate spheroidal mass is used with the angular coordinates kept constant. This gives us a new expression for gravitational length contraction. As an illustration of this gravitational phenomenon in oblate spheroidal gravitational fields, we consider a long stick lying "radially" along the equator in the approximate gravitational field of a static homogenous oblate spheroidal mass such as the Earth and we let the ξ-coordinates of the ends be ξ_1 and ξ_2, where $\xi_2 > \xi_1$. With this, we find the expression for its proper length and deduce that the length is reduced in the gravitational field. This computation affirms the soundness of our extension and confirms the assertion from Schwarzschild's metric that not only is length contracted in gravitational fields but space also.

Gravitational spectral shift of light. We consider a beam of light (photons) moving from a source or emitter at a fixed point in the gravitational field of the oblate spheroidal body to an observer or receiver at a fixed point in the same gravitational field. Einstein's equation of motion for a photon is used to derive an expression for the shift in frequency of a photon moving in the gravitational field of an oblate spheroidal mass. We then as an illustration of the expression obtained, consider a signal of light emitted and received along the equator of the static homogenous oblate spheroidal Earth (in the approximate gravitational field). The ratio of the shift in frequency to the frequency of the emitted light at various points in the equatorial plane and received on the equator at the surface of the static homogeneous oblate spheroidal Earth is computed using our derived equation. Also, the ratio of the shift in frequency of light to the frequency of the emitted light on the equator at the surface and received at various points along the equator of the static homogeneous oblate spheroidal Earth is also computed. It is worth noting that we deliberately used emitters and receivers at rest in this gravitational field to avoid shifts in frequency due to Doppler effect. However, in more practical cases, the gravitational spectral shift is always compounded with the special relativistic shift (Doppler shift). This yields a general expression for the shift in frequency when there is a relative motion between the emitter and receiver.

Chapter 3. Results and Discussion

§3.1. General relativistic mechanics in homogeneous oblate spheroidal gravitational fields

§3.1.1. Motion of particles of non-zero rest masses in homogeneous oblate spheroidal space-time

The contravariant metric tensor $g^{\mu\nu}$ for this gravitational field is obtained as

$$g^{00} = \left(1 + \frac{2}{c^2}\, f(\eta,\, \xi)\right)^{-1}, \tag{3.1}$$

$$g^{11} = -\frac{\left(1-\eta^2\right)\left(1+\xi^2-\eta^2\right)\left[\eta^2\left(1-\eta^2\right) + \frac{\xi^2\left(1+\xi^2\right)}{1+\frac{2}{c^2}f(\eta,\xi)}\right]}{\frac{a^2}{1+\frac{2}{c^2}f(\eta,\xi)}\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\right]^2}, \tag{3.2}$$

$$g^{12} = g^{21} = -\frac{\eta\xi\left(1-\eta^2\right)\left(1+\xi^2\right)\left(1+\xi^2-\eta^2\right)}{\frac{a^2}{1+\frac{2}{c^2}f(\eta,\xi)}\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\right]^2} \times \\ \times\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right], \tag{3.3}$$

$$g^{22} = -\frac{\left(1+\xi^2\right)\left(1+\xi^2-\eta^2\right)\left[\xi^2\left(1+\xi^2\right)+\frac{\eta^2\left(1-\eta^2\right)}{1+\frac{2}{c^2}f(\eta,\xi)}\right]}{\frac{a^2}{1+\frac{2}{c^2}f(\eta,\xi)}\left[\eta^2\left(1-\eta^2\right)+\xi^2\left(1+\xi^2\right)\right]^2}, \tag{3.4}$$

$$g^{33} = \left[a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\right]^{-1}. \tag{3.5}$$

The contravariant metric tensor has two additional non-zero components not found in Schwarzschild field. Notice that unlike in Schwarzschild field; where all the non-zero components of the contravariant tensor are simply reciprocals of the covariant metric tensor; only equations (3.1) and (3.5) are reciprocals of their respective covariant tensors. The other non-zero components have a common denominator. The coefficients of affine connection found have fourteen non zero components., dependent on a single unknown function f. Schwarzschild's connection coefficients on the other hand have ten non-zero components dependent on the gravitational scalar potential exterior to the spherically symmetric mass [20].

Using the general relativistic equation of motion for test particles and the coefficients of affine connection for the gravitational field exterior to an oblate spheroidal mass the following equations of motion are obtained. The time equation of motion is obtained as

$$\frac{d}{d\tau}(\ln\dot{t}) + \frac{d}{d\tau}\left[\ln\left(1+\frac{2}{c^2}f(\eta,\,\xi)\right)\right] = 0 \tag{3.6}$$

with solution as

$$\dot{t} = A\left(1+\frac{2}{c^2}f(\eta,\,\xi)\right)^{-1}. \tag{3.7}$$

As $t \to \tau$, $f(\eta,\,\xi) \to 0$ and the constant $A \equiv 1$. Thus,

$$\dot{t} = \left(1+\frac{2}{c^2}f(\eta,\,\xi)\right)^{-1}. \tag{3.8}$$

Equation (3.8) is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that in Schwarzschild's gravitational field. Interestingly, our expression differs greatly from that obtained by [21]. In his case, he obtains $\dot{t}$ as an exponential function dependent on his unknown function $F(\xi)$. Thus, our expression in its merit stands out uniquely, as an extension of the results in Schwarzschild's field. Also, it tends out most remarkably that our unknown function can be evaluated from the gravitational field equations.

The η equation of motion is

$$\ddot{\eta} + \Gamma^{1}_{00}\, c^2 \dot{t}^2 + \Gamma^{1}_{11}\, \dot{\eta}^2 + \Gamma^{1}_{22}\, \dot{\xi}^2 + \Gamma^{1}_{33}\, \dot{\phi}^2 + 2\Gamma^{1}_{12}\, \dot{\eta}\, \dot{\xi} = 0\,. \tag{3.9}$$

The ξ equation of motion is given as

$$\ddot{\xi} + \Gamma^{1}_{00}\, c^2 \dot{t}^2 + \Gamma^{1}_{11}\, \dot{\eta}^2 + \Gamma^{1}_{22}\, \dot{\xi}^2 + \Gamma^{1}_{33}\, \dot{\phi}^2 + 2\Gamma^{1}_{12}\, \dot{\eta}\, \dot{\xi} = 0\,. \tag{3.10}$$

The azimuthal equation of motion is obtained as

$$\dot{\phi} = \frac{l}{(1-\eta^2)\,(1+\xi^2)}\,, \tag{3.11}$$

where l is a constant of motion. Herein l physically corresponds to the angular momentum and hence equation (3.11) is the law of conservation of angular momentum in this gravitational field. It does not depend on the gravitational potential and is of same form as that obtained in Schwarzschild's and Newton's dynamical theory of gravitation. It is worth emphasizing that although the form is the same, it stands out unique as the parameters are in oblate spheroidal coordinates.

§3.1.2. Planetary motion and motion of photons in the equatorial plane of homogeneous oblate spheroidal gravitational fields

The Lagrangian, in the space-time exterior to an oblate spheroid can be written explicitly in oblate spheroidal coordinates as

$$L = \frac{1}{c}\left[-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{d\eta}{d\tau}\right)^2 - 2\,g_{12}\left(\frac{d\eta}{d\tau}\right)\left(\frac{d\xi}{d\tau}\right) - \right.$$
$$\left. - g_{22}\left(\frac{d\xi}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2 \right]^{1/2}. \tag{3.12}$$

For orbits in the equatorial plane of a homogeneous oblate spheroidal mass; $\eta \equiv 0$ and using the Lagrangian it is shown (using the fact that the gravitational field is a conservative field) that the law of conservation of energy in the equatorial plane of the gravitational field exterior to an oblate spheroidal mass is

$$\left(1 + \frac{2}{c^2}\, f(\xi)\right) \dot{t} = k\,, \qquad \dot{k} = 0\,, \tag{3.13}$$

where k is a constant. Notice that this equation is exactly the same as the expression obtained from the time equation of motion for test particles. This expression has never been obtained before. Thus, our use of the metric tensor and Lagrangian mechanics in oblate spheroidal gravitational field yields the first ever documented expression for the conservation of energy in this field [9].

Also, the law of conservation of angular momentum in the equatorial plane of the gravitational field exterior to an oblate spheroidal body is obtained as

$$\left(1+\xi^2\right)\dot{\phi}=l\,, \qquad \dot{l}=0\,, \tag{3.14}$$

where l is a constant. It is interesting and instructive to note that this expression is equivalent to that obtained from the general relativistic azimuthal equation of motion for test particles in the gravitational field exterior to an oblate spheroidal mass. Thus our method for obtaining the laws of conservation of total energy and angular momentum in this section is mathematically more convenient and physically more interesting than the method in the previous section. Instead of going through the rigorous tensor analysis to derive the affine connections before proceeding to derive the conservation laws; we simply need to build the covariant metric tensor and use the generalized Lagrangian to deduce the conservation laws.

Using the fact that the Lagrangian $L=\epsilon$, with $\epsilon=1$ for time like orbits and $\epsilon=0$ for null orbits the planetary equation of motion in this gravitational field is

$$\begin{aligned}\frac{d^2u}{d\phi^2} &- 3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{u+u^2}{2}\left(u^2-u+2\right)\times\\ &\times\left(1+\frac{2}{c^2}\,f(u)\right)=\left(\frac{1+u^2}{acl}\right)^2\left(a^2c^2u^2-1-u^2\right)\frac{d}{du}f(u)\,.\end{aligned} \tag{3.15}$$

It can be solved to obtain the perihelion precision of planetary orbits. This is opened up for further research.

The photon equation of motion in the vicinity of a static massive homogenous oblate spheroidal body is obtained as

$$\begin{aligned}\frac{d^2u}{d\phi^2} &- 3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{u+u^2}{2}\left(u^2-u+2\right)\times\\ &\times\left(1+\frac{2}{c^2}\,f(u)\right)=\frac{u^2}{c^2}\left(1+u^2\right)^2\frac{d}{du}\,f(u)\,.\end{aligned} \tag{3.16}$$

In the limit of special relativity, some terms in equation (3.16) vanish and the equation becomes

$$\frac{d^2u}{d\phi^2}-3u\left(1+u^2\right)\frac{du}{d\phi}+\frac{u+u^2}{2}\left(u^2-u+2\right). \tag{3.17}$$

The solution of the special relativistic case, equation (3.17) can be used to solve the general relativistic equation, (3.16). This can be done by taking the general solution of equation (3.17) to be a perturbation of the solution of equation (3.16). The immediate consequence of this analysis is that it will produce a new expression for the total deflection of light grazing a massive oblate spheroidal body such as the Sun. This is also open for further research and astrophysical interpretations.

§3.1.3. Effects of oblateness of the Sun and planets on some gravitational phenomena

Gravitational scalar potential along the pole and equator of the homogeneous oblate spheroidal Sun and planets. The computed numerical values of the constants ξ_0 and a for the oblate spheroidal bodies in the Solar System are given in Table 2.

Body	Equatorial radius $x_0 \times 10^3$	Polar radius $z_0 \times 10^3$	ξ_0	a, m
Sun	700,00	699,994	241.52	2.89829×10^6
Mercury	2,440	2,440	—	—
Venus	6,052	6,052	—	—
Earth	6,378	6,356	12.01	5.29226×10^5
Mars	3,396	3,376	09.17	3.68157×10^5
Jupiter	71,490	66,843	02.64	2.53193×10^7
Saturn	60,270	53,761	01.97	2.72899×10^7
Uranus	25,560	24,793	03.99	6.21378×10^6
Neptune	24,760	24,116	04.30	5.60837×10^6

Table 2: Computed constants ξ_0 and a for the Sun and planets [10].

The gravitational scalar potential exterior to a homogeneous oblate spheroid [4] is given as

$$f(\eta,\,\xi) = B_0\,Q_0(-i\xi) + B_2\,Q_2(-i\xi)\,P_2(\eta)\,, \tag{3.18}$$

where Q_0 and Q_2 are the Legendre functions linearly independent to the Legendre polynomials P_0 and P_1 respectively. B_0 and B_2 are constants with approximate expressions as

$$B_0 \approx i\,\frac{4\pi G\rho_0 a^2\xi_0^5}{3\,(1+\xi_0^2)}\,, \tag{3.19}$$

$$B_2 \approx i\,\frac{4\pi G\rho_0 a^2\xi_0^5}{3\big[44\xi_0^2 + (1+3\xi_0^2)\binom{2}{0}\big]}\,. \tag{3.20}$$

The mean density ρ_0 for various bodies in the universe is taken according to the astronomical data. With the values of ξ_0 and a in Table 2, we get the values for the constants B_0 and B_2 for the homogeneous oblate spheroidal Sun and planets as in Table 3.

Body	Mean density ρ_0, kg/m^3	$i\,B_0$, Nm/kg	$i\,B_2$, Nm/kg
Sun	1409	4.67961×10^{13}	8.91380×10^{7}
Mercury	5400	—	—
Venus	5200	—	—
Earth	5500	7.43766×10^{8}	1.70123×10^{5}
Mars	3900	1.13049×10^{8}	4.40357×10^{5}
Jupiter	1300	3.76352×10^{9}	1.50951×10^{7}
Saturn	690	8.76690×10^{8}	5.70607×10^{6}
Uranus	1300	8.41939×10^{8}	1.61800×10^{6}
Neptune	1600	1.06534×10^{9}	1.78225×10^{6}

Table 3: Values of the constants B_0 and B_2 for the Sun and Planets [17].

By considering the first two terms of the series expansion of the Legendre functions, we can write

$$f(\eta,\,\xi) \approx \frac{B_0}{3\xi^3}\left(1+3\xi^2\right) i + \frac{B_2}{30\xi^3}\left(7+15\xi^2\right) i\,, \tag{3.21}$$

$$f(\eta,\,\xi) \approx \frac{B_0}{3\xi^3}\left(1+3\xi^2\right) i - \frac{B_2}{30\xi^3}\left(7+15\xi^2\right) i\,, \tag{3.22}$$

as the respective expressions for the gravitational scalar potential along the equator and pole exterior to homogeneous oblate spheroidal bodies. Now, with the computation of the constant ξ_0 for the homogeneous oblate spheroidal Sun and planets, we can now evaluate the scalar potential along the equator and the pole at various points (multiples of ξ_0) exterior to the Sun and planets. The detailed results are presented in [10]. Our computations agree satisfactorily with the experimental fact that the Gravitational Scalar Potential exterior to any regularly shaped object has maximum magnitude on the surface of the body and decreases to zero at infinity.

The consequence of the results obtained above is that the exact shape of the planets and Sun was used to obtain the gravitational scalar potential on the surface at the pole and equator. Thus, instead of using the values obtained by considering the Sun and planets as homogeneous spheres, our experimentally convenient values obtained can now be used.

The door is now open for the computation of values for various gravitational phenomena exterior to the static homogeneous oblate spheroidal Sun and planets along the equator and pole. Some of these phenomena include gravitational length contraction and time dilation.

Gravitational time dilation in fields exterior to static oblate spheroidal distributions of mass. Consider a clock at rest at a fixed point (η, ξ, ϕ) in the gravitational field exterior to an oblate spheroidal mass, the world line element for this gravitational field reduces to

$$dt = \left(1 + \frac{2}{c^2}\, f(\eta, \xi)\right)^{1/2} d\tau\,. \tag{3.23}$$

Expanding the right hand side gives

$$dt = \left(1 + \frac{2}{c^2}\, f(\eta, \xi) + \dots\right)^{1/2} d\tau\,. \tag{3.24}$$

We obtain that $dt > d\tau$ (dilation). Thus, coordinate time of a clock in this gravitational field is dilated relative to proper time.

As an illustration, consider two events at fixed points exterior to the homogenous oblate spheroidal Earth along the equator, separated in this gravitational field by coordinate time dt and proper time $d\tau$. Substituting the values for the gravitational scalar potential into the equation for gravitational time dilation (3.24), (approximate fields) yields the results presented in Table 4.

Thus, we conclude that clock runs more slowly at a smaller distance from the massive oblate spheroidal body. In other words, clocks will run slower at lower gravitational potentials (deeper within a gravity well). This was first confirmed experimentally in the laboratory by the Hafele-Keating experiment [22]. Today, there are numerous direct measurements of gravitational time dilation using atomic clocks [23], while ongoing validation is provided as a side-effect of the operation of Global Positioning System (GPS). One important experiment that was conducted to support Einstein's principle of time dilation was the experiment by Rossi and Hall in 1941 and repeated recently in accelerator rings. In this experiment, muons travelling with a velocity close to the velocity of light are observed to survive longer than muons that travel with velocities that are much less than that of light. Also, in 1976, the Smithsonian Astrophysical Observatory sent aloft a Scout rocket to an altitude of 10,000 km. This expedition also confirmed gravitational time dilation.

Fixed point along the Equator	Radial distance along the Equator, km	dt as a factor of $d\tau$
ξ_0	6,378	1.306170
$2\,\xi_0$	12,723	1.122655
$3\,\xi_0$	19,075	1.076871
$4\,\xi_0$	25,430	1.055996
$5\,\xi_0$	31,784	1.044042
$6\,\xi_0$	38,140	1.036296
$7\,\xi_0$	44,495	1.030867
$8\,\xi_0$	50,851	1.026852
$9\,\xi_0$	57,207	1.023761
$10\,\xi_0$	63,562	1.021308

Table 4: Coordinate time at fixed points along the equator in the gravitational field exterior to the Earth as a factor of proper time [11].

Gravitational length contraction in fields exterior to oblate spheroidal distributions of mass. Here, the space part of the world line element in the gravitational field exterior to an oblate spheroidal mass is used with the angular coordinates kept constant. This gives us an expression for gravitational length contraction in this field as

$$ds = \left(\frac{a^2}{1+\xi^2-\eta}\left[\frac{\xi^2}{1+\frac{2}{c^2}\,f(\eta,\xi)} + \frac{\eta^2\left(1-\eta^2\right)}{\left(1+\xi^2\right)}\right]\right)^{1/2} d\xi\,. \qquad (3.25)$$

Along the equatorial line, $\eta = 0$ and equation becomes

$$ds = a\,\xi\left(1+\xi^2\right)^{-1/2}\left(1+\frac{2}{c^2}\,f(\eta,\,\xi)\right)^{-1/2} d\xi\,. \qquad (3.26)$$

It can be shown that $ds > d\xi$ from equation (3.26). In other words, the coordinate distance separating these two points is contracted in this gravitational field. Thus, we can write

$$d\xi = (a\,\xi)^{-1}\left(1+\xi^2\right)^{1/2}\left(1+\frac{2}{c^2}\,f(\eta,\,\xi)\right)^{1/2} ds \qquad (3.27)$$

as our expression for gravitational length contraction along the equator in this gravitational field.

As an illustration of this gravitational phenomenon, we can consider a long stick lying radially along the equator in the approximate gravitational field of a static homogeneous oblate spheroidal mass such as the

Earth. Let the ξ-coordinates of the ends be ξ_1 and ξ_2, where $\xi_2 > \xi_1$. Then the formula for its proper length will be as that found in [11].

Gravitational spectral shift in gravitational fields exterior to oblate spheroidal distributions of mass or pressure Here, we consider a beam of light moving from a source or emitter at a fixed point in the gravitational field of the oblate spheroidal body to an observer or receiver at a fixed point in the same gravitational field. Einstein's equation of motion for a photon is used to derive an expression for the shift in frequency of a photon moving in the gravitational field of an oblate spheroidal mass as.

Now, consider a beam of light moving from a source or emitter (E) at a fixed point in the gravitational field of an oblate spheroidal body to an observer or receiver (R) at a fixed point in the field. Let the space-time coordinates of the emitter and receiver be t_{E}, η_{E}, ξ_{E}, ϕ_{E} and t_{R}, η_{R}, ξ_{R}, ϕ_{R} respectively. It is a well known fact that light moves along a null geodesic given by

$$d\tau = 0\,. \tag{3.28}$$

Thus, the world line element for a photon (light) takes the form

$$c^2 g_{00}\, dt^2 = g_{11}\, d\eta^2 + 2\, g_{12}\, d\eta\, d\xi + g_{22}\, d\xi^2 + g_{33}\, d\phi^2. \tag{3.29}$$

Substituting the covariant metric tensor for this gravitational field and let u be a suitable parameter that can be used to study the motion of a photon in this gravitational field then equation (3.29) can be written as

$$\frac{dt}{du} = \frac{1}{c}\left(1 + \frac{2}{c^2}\, f(\eta,\, \xi)\right)^{-1/2} ds\,, \tag{3.30}$$

where ds is defined as

$$\begin{aligned} ds^2 = &-\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2\left(1+\frac{2}{c^2}\, f(\eta,\,\xi)\right)^{-1} + \frac{\xi^2\left(1+\xi^2\right)}{\left(1-\eta^2\right)}\right]\left(\frac{d\eta}{du}\right)^2 - \\ &- \frac{2\,a^2\eta\,\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2} f(\eta,\,\xi)\right)^{-1}\right]\frac{d\eta}{du}\frac{d\xi}{du} - \\ &- a^2\left(1+\xi^2\right)\left(1-\eta^2\right)\left(\frac{d\phi}{du}\right)^2 - \\ &- \frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}\, f(\eta,\,\xi)\right)^{-1} + \frac{\eta^2\left(1+\eta^2\right)}{\left(1-\xi^2\right)}\right]\left(\frac{d\xi}{du}\right)^2. \end{aligned} \tag{3.31}$$

Integrating equation (3.30) for a signal of light moving from emitter to receiver gives

$$t_{\mathrm{R}} - t_{\mathrm{E}} = \frac{1}{c}\int_{u_{\mathrm{E}}}^{u_{\mathrm{R}}} \left[\left(1 + \frac{2}{c^2}\, f(\eta,\, \xi)\right)^{-1/2} ds\right] du\,. \tag{3.32}$$

The time interval between emission and reception of all light signals is well known to be the same for all light signals in relativistic mechanics (constancy of the speed of light) and thus the integral on the right hand side is the same for all light signals. Consider two light signals designated 1 and 2 then

$$\Delta t_{\mathrm{R}} = \Delta t_{\mathrm{E}}\,. \tag{3.33}$$

Hence, coordinate time difference of two signals at the point of emission equals that at the point of reception. From our expression for gravitational time dilation in this gravitational field, we can write

$$\Delta\tau_{\mathrm{R}} = \left(1 + \frac{2}{c^2}\, f(\eta,\, \xi)\right)^{1/2} \Delta t_{\mathrm{R}} \tag{3.34}$$

Hence

$$\frac{\Delta\tau_{\mathrm{R}}}{\Delta\tau_{\mathrm{E}}} = \left(\frac{1 + \frac{2}{c^2}\, f_{\mathrm{R}}(\eta,\xi)}{1 + \frac{2}{c^2}\, f_{\mathrm{E}}(\eta,\xi)}\right)^{1/2}. \tag{3.35}$$

Now, consider the emission of a peak or crest of light wave as one event. Let n be the number of peaks emitted in a proper time interval $\Delta\tau_{\mathrm{E}}$, then, by definition, the frequency of the light relative to the emitter, ν_{E}, is given as

$$\nu_{\mathrm{E}} = \frac{n}{\Delta\tau_{\mathrm{E}}}\,. \tag{3.36}$$

Similarly, since the number of cycles is invariant, the frequency of light relative to the receiver, ν_{R}, is given as

$$\nu_{\mathrm{R}} = \frac{n}{\Delta\tau_{\mathrm{R}}}\,. \tag{3.37}$$

Consequently,

$$\frac{\nu_{\mathrm{R}}}{\nu_{\mathrm{E}}} = \frac{\Delta\tau_{\mathrm{E}}}{\Delta\tau_{\mathrm{R}}} = \left(\frac{1 + \frac{2}{c^2}\, f_{\mathrm{E}}(\eta,\xi)}{1 + \frac{2}{c^2}\, f_{\mathrm{R}}(\eta,\xi)}\right)^{1/2}. \tag{3.38}$$

The expressions on the right hand side of equation (3.38) are converging and can be expanded binomially in approximate gravitational fields. This gives

$$z \equiv \frac{\Delta\nu}{\nu_{\rm E}} \equiv \frac{\nu_{\rm R} - \nu_{\rm E}}{\nu_{\rm E}} \approx \frac{1}{c^2}\big(f_{\rm E}\,(\eta,\,\xi) - f_{\rm R}\,(\eta,\,\xi)\big) \tag{3.39}$$

to the order of c^{-2}. It follows from equation (3.39) that if the source is nearer the body than the receiver then $f_{\rm E}\,(\eta,\,\xi) - f_{\rm R}\,(\eta,\,\xi)$ and hence $\Delta\nu < 0$. This indicates that there is a reduction in the frequency of light when the source or emitter is nearer the body than the receiver. The light is said to have undergone a red shift (that is the light moves towards red in the visible spectrum). Otherwise (source further away from body than receiver), the light undergoes a blue shift.

This was experimentally confirmed in the laboratory by the Pound-Rebka experiment in 1959 [24] (they used the Mossbauer effect to measure the change in frequency in gamma rays as they travelled from the ground to the top of Jefferson Labs at Havard University). The effect of a gravitational potential difference on the apparent energy of the 14.4 keV gamma ray of Fe^{57} was found by Pound and Rebka [25] to agree within uncertainties, with Einstein's prediction based on his principle of equivalence (General Relativity). Pound and Rebka in 1964 [26] improved on their earlier results confirming Einstein's prediction to greater precision. The resonance of the 14.4 keV Fe^{57} gamma ray between Iron foils was still employed. The same height as in the earlier experiment in the Jefferson Physical Laboratory (22.5 m) was also used. This gravitational phenomenon was later confirmed by astronomical observations.

Now, suppose the Pound-Rebka experiment was performed at the surface of the Earth on the equator. Then, since the gamma ray frequency shift was observed at a height of 22.5 m above the surface, we model our theoretical computation and calculate the theoretical value for this shift.

Recall that at the surface of the Earth, on the equator, we have $x_0 = 6378000$ m. Numerical values of a and ξ_0 are defined as in Table 2. The value of x at a height of 22.5 m above the surface is trivially $x_0 + 22.5 = 6378022.5$ m. Using the value of a for the Earth from Table 2 it is shown that ξ at the point is 12.0100447. For spectral shift of light emitted at the surface and received 22.5 m above the surface of the Earth, along the equator, equation (3.39) holds in approximate gravitational fields. In this case, $f_{\rm E} = -\,6.2079113\times10^7$ Nm/kg; this is the gravitational scalar potential on the surface of the Earth on the equator. At the reception point, we use the value of ξ and compute $f_{\rm R} = -\,6.207888\times10^7$ Nm/kg. Thus, substituting the values of $f_{\rm E}$, $f_{\rm R}$ and c into equation (3.39) yields the shift in frequency as

$z \simeq 2.578 \times 10^{-15}$. This value is quite close to that obtained by Pound and Rebka ($z \simeq 2.45 \times 10^{-15}$) in 1964. The closeness of our theoretically computed value for the Pound-Rebka experiment is remarkable indeed. The difference can be accounted for by the slight discrepancy between theory and experiment. Approximations made to the gravitational scalar potential are also a possible contributing factor.

We can now conveniently predict the gravitational spectral shift for Pound-Rebka experiment, if it was performed along the equator of the Sun and oblate spheroidal planets. As in the case of the Earth, it can be shown that the predicted shift in frequency is as shown in Table 5.

Body	Distance km	ξ	f_{R} Nm/kg	Predicted shift
Sun	700,022.5	241.527	$-1.9373218 \times 10^{11}$	-2.85889×10^{-21}
Mars	3418.5	9.231	-1.2317966×10^{7}	-9.24256×10^{-20}
Jupiter	71512.5	1.971	-1.4958977×10^{9}	$-1.010111 \times 10^{-20}$
Saturn	60292.5	1.971	-4.8484869×10^{8}	$-1.902222 \times 10^{-20}$
Uranus	25582.5	3.994	-2.1522082×10^{8}	$-4.647889 \times 10^{-20}$
Neptune	24782.5	4.304	-2.5196722×10^{8}	$-5.168667 \times 10^{-20}$

Table 5: Predicted Pound-Rebka shift in frequency for the Sun and other oblate spheroidal planets.

With these predictions, astrophysicists and astronomers can now attempt carrying out similar experiments on these planets. Although, the prospects of carrying out such experiments on the surface of some of the planets and Sun are less likely (due to temperatures on their surfaces and other factors); theoretical studies of this type helps us to understand the behavior of photons as they leave or approach these astrophysical bodies. This will thus aid in the development of future astronomical instruments that can be used to study these heavenly bodies.

Also, our expression for gravitational time dilation and spectral shift can be used in place of those obtained from Schwarzschild's field in the expression of relativistic effects in the GRACE satellites. The GRACE mission consists of two identical satellites orbiting the Earth at an altitude of about 500 km. Dual-frequency carrier-phase GPS receivers are flying on both satellites. They are used for precise orbit determination and to time-tag the K-band ranging systems used to measure changes in the distance between the two satellites. Kristine et al. [27] developed an expression for the relativistic effects of low Earth orbiters (the GRACE satellites). Their expression can be re-modified by considering our derived expressions in this work. This will improve on the accuracy of GRACE data.

§3.2. General relativistic mechanics in homogeneous prolate spheroidal gravitational fields

§3.2.1. Gravitational field equations exterior to a homogeneous prolate spheroidal mass

The generalized covariant metric tensor exterior to static homogeneous prolate spheroidal distributions of mass or pressure is given as equations (2.29) to (2.33). The contravariant metric tensor for this gravitational field $g^{\mu\nu}$ can be obtained with the aid of the tensor equations (2.34) and (2.35). The contravariant metric tensor has two additional non-zero components not found in Schwarzschild field.

The coefficients of affine connection for the gravitational field exterior to a static homogeneous prolate spheroidal mass can be found. The curvature tensor for this gravitational field has twenty four non-zero components.

The Ricci tensor for this gravitational field can thus be composed in terms of the curvature tensor and the curvature scalar, R, can also be obtained. The general relativistic field equations for a region exterior to any astrophysical body are given as

$$R_{00} - \frac{1}{2}\,R\,g_{00} = 0\,, \tag{3.40}$$

$$R_{11} - \frac{1}{2}\,R\,g_{11} = 0\,, \tag{3.41}$$

$$R_{12} - \frac{1}{2}\,R\,g_{12} = 0\,, \tag{3.42}$$

$$R_{22} - \frac{1}{2}\,R\,g_{22} = 0\,, \tag{3.43}$$

$$R_{33} - \frac{1}{2}\,R\,g_{33} = 0\,. \tag{3.44}$$

The gravitational field equations derived are second order partial differential equations that can be solved and interpreted. All its mathematically possible solutions may then be distinguished by physical considerations, such as consistency with astrophysical or astronomical observations, data and facts. Hence, in principle, our arbitrary function, $f(\eta,\,\xi)$, which uniquely and completely determines the solution of Einstein's gravitational metric tensor field exterior to the static homogeneous prolate spheroidal mass or pressure distributions can be found.

It is interesting to note that the number of distinct non-zero components of the Ricci tensor is five. The number of distinct non-zero components of the Ricci tensor is always the same, no matter the nature of the mass distribution within prolate spheroidal regions. This number corresponds to the number of distinct non-zero components of the metric tensor in this field. It is also equal to the number of distinct field equations possible in the gravitational field.

Thus, generally, in prolate spheroidal fields, the rigorous field equations are nonlinear differential equations. The Schwarzschild's solution is a rigorous solution of Einstein's field equations and we have succeeded to extend his results to fields exterior to prolate spheroidal masses. Schwarzschild's solution is significant because it is the only solution of the field equations in empty space which is static, which has spherical symmetry, and which goes over into the flat metric at infinity [1]. Also, in fields exterior to static homogenous prolate spheroidal masses (with the approximate expression for our arbitrary function given as Newton's gravitational scalar potential exterior to the body), the metric reduces conveniently to the flat space metric for prolate spheroidal masses at infinity (since the gravitational potential reduces to zero at infinity).

§3.2.2. Solutions to gravitational field equations exterior to homogeneous prolate spheroidal masses

It can be shown trivially that no two of these five Einstein field equations possess a common simultaneous solution. Consequently these equations may only be solved separately and their different solutions applied whenever and wherever necessary and useful in physical theories.

It is also obvious that in the case of the static homogenous distribution of mass within a prolate spheroidal region in this research work, all the five nontrivial Einstein field equations possess their own different solutions which may be applied whenever and wherever useful in physical theory.

In this section, we construct the solution for the first field equation, equation (3.40). Writing the various terms of the field equation (3.40) explicitly in terms of the metric tensor and simplifying by grouping yields a more explicit expression of the field equation with only terms of order c^{-2} as

$$K_1(\eta,\xi)\, f_{\eta\eta} + K_2(\eta,\xi)\, f_{\eta\xi} + K_3(\eta,\xi)\, f_{\xi\xi} + \\ + K_4(\eta,\xi)\, f_{\eta} + K_5(\eta,\xi)\, f_{\xi} + K_6(\eta,\xi)\, f = 0\,, \qquad (3.45)$$

where the coefficients K_i $(i=1,\dots,6)$ are functions of ξ and η only.

Equation (3.45) is thus our simplified exterior field equation to the order of c^{-2} for homogeneous prolate spheroidal gravitational fields. We can now conveniently seek to construct the astrophysically most satisfactory solutions of equation (3.45) which are convergent in the exterior space-time:

$$\xi > \xi_0 \quad \text{and} \quad -1 \leqslant \eta \leqslant 1\,. \tag{3.46}$$

Let us now seek the solution f, of our field equation (3.45) in the form of a power series

$$f\left(\eta,\xi\right) = \sum_{n=1}^{\infty} R_n\left(\xi\right)\eta^n, \tag{3.47}$$

where R_n is a function to be determined for each value of $n = 0, 1, 2, \ldots$ Substituting equation (3.47) into (3.45) and using the fact that $\{\eta^n\}_{n=0}^{\infty}$ is a linearly independent set, we can equate the coefficients of η^n on both sides and hence obtain the equations satisfied by the functions R_n. From the coefficients of η^0 we obtain the equation

$$\begin{aligned} &\xi^2\left(\xi^2-1\right)R_1^{'}\left(\xi\right) + 2\left(\xi^2-1\right)\left(\xi^3-\xi-2\right)R_1\left(\xi\right) + \left(\xi^2-1\right)^2 \times \\ &\times \left(4\xi^3+2\xi^2+\xi-1\right)R_0^{\prime}\left(\xi\right) + \Big[4\left(2+\xi^2\right) + 8\xi^8\left(\xi^2-1\right) + \\ &+ 2\left(\xi^2-1\right)^2\left(2\xi^4-2\xi^8-4\xi^3-1\right)\Big]R_0\left(\xi\right) = 0\,. \end{aligned} \tag{3.48}$$

Equation (3.48) is the first recurrence differential equation for the unknown functions R_n. Similarly all the other recurrence differential equations follow. There are infinitely many of the recurrence differential equations to determine all the unknown functions.

Firstly, it is most interesting and instructive to note that according to the first recurrence differential equation (3.48), the unknown functions R_0 and R_1 are actually arbitrary. Therefore we have the freedom to choose them to satisfy the physical requirements or needs of any particular distribution or area of application. Thus, we realize that they can be chosen in such a way that there are generalizations of the gravitational scalar potential exterior to the mass distribution.

Secondly, we note that the first recurrence differential equation (3.48) determines the unknown function R_1 in terms of R_0. Similarly, the other recurrence differential equations will determine all the other unknown functions $R_2, \ldots,$ in terms of R_0. Hence we obtain the general exterior solution of equation (3.45) in terms of R_0. This is our mathematically most simple and astrophysically most satisfactory general exterior solution of order c^{-2}.

§3.2.3. Motion of particles of non-zero rest masses exterior to static homogeneous prolate spheroidal space-time

The time equation of motion is obtained as

$$\frac{d}{d\tau}\left(\ln \dot{t}\right) + \frac{d}{d\tau}\left[\ln\left(1 + \frac{2}{c^2}\, f\left(\eta, \xi\right)\right)\right] = 0 \tag{3.49}$$

with solution

$$\dot{t} = \left(1 + \frac{2}{c^2}\, f\left(\eta, \xi\right)\right)^{-1}. \tag{3.50}$$

Equation (3.50) is the expression for the variation of the time on a clock moving in this gravitational field. It is of same form as that obtained in the oblate spheroidal gravitational field and in Schwarzschild's field The η-equation and ξ-equation of motion are

$$\ddot{\eta} + \Gamma^1_{00}\, c^2\, \dot{t}^2 + \Gamma^1_{11}\, \dot{\eta}^2 + \Gamma^1_{22}\, \dot{\xi}^2 + \Gamma^1_{33}\, \dot{\phi}^2 + 2\Gamma^1_{12}\, \dot{\eta}\dot{\xi} = 0\,, \tag{3.51}$$

$$\ddot{\xi} + \Gamma^2_{00}\, c^2\, \dot{t}^2 + \Gamma^2_{11}\, \dot{\eta}^2 + \Gamma^2_{22}\, \dot{\xi}^2 + \Gamma^2_{33}\, \dot{\phi}^2 + 2\,\Gamma^2_{12}\, \dot{\eta}\dot{\xi} = 0\,. \tag{3.52}$$

For azimuthal motion,

$$\frac{d}{d\tau}\left(\ln \dot{\phi}\right) + \frac{d}{d\tau}\left[\ln\left(\eta^2 - 1\right)\left(1 - \xi^2\right)\right] = 0\,, \tag{3.53}$$

with solution

$$\dot{\phi} = \frac{l}{\left(\eta^2 - 1\right)\left(1 - \xi^2\right)}\,, \tag{3.54}$$

where l is a constant of motion. Herein l physically corresponds to the angular momentum. This is the law of conservation of angular momentum in this gravitational field. It has the same form as that obtained in the oblate spheroidal gravitational field and does not depend on the gravitational potential. Therefore, it is of same form as that obtained in Schwarzschild's and Newton's dynamical theory of gravitation. The significance of these results is that the law of conservation of angular momentum takes the same form in the three different gravitational fields and thus the expression for this law of mechanics is invariant with respect to the three gravitational fields.

§3.2.4. Orbits in homogeneous prolate spheroidal space-time

The Lagrangian in the space-time exterior to a prolate spheroid is used to obtain

$$\frac{d^2u}{d\phi^2} - \frac{2u}{1+u^2}\frac{du}{d\phi} + \left(\frac{1+u^2}{acl}\right)^2 \frac{df}{du} = 0 \tag{3.55}$$

as the planetary equation of motion and

$$\frac{d^2u}{d\phi^2} - \frac{2u}{1+u^2}\frac{du}{d\phi} = 0 \tag{3.56}$$

as the photon equation of motion in the vicinity of a static massive homogenous prolate spheroidal body.

Conclusion

The practicability of the findings in this work is an encouraging factor. More so, that in this age of computational precision, the applications of these results is another factor. This work exposes the philosophical and theoretical successes/failures of General Relativity theory to the advancement of studies in gravitation. The astrophysical applications of our extension abound as all applications of Schwarzschild's metric in studying gravitational phenomena in the Solar System can now be studied using our new approach.

With the formulation of our mathematically most simple and astrophysically most satisfactory solutions to Einstein's gravitational field equations the way is opened for the solution of the general relativistic equations of motion for all test particles in the gravitational fields of all static homogeneous distributions of mass within prolate spheroidal regions in the universe. And precisely because these equations contain the pure Newtonian as well as post-Newtonian gravitational scalar potentials all their predictions shall be most naturally comparable to the corresponding predictions from the pure Newtonian theory. This is most satisfactory indeed.

It is now obvious how our work may be emulated to

1) Derive a mathematically most simple structure for all the metric tensors in the space-times exterior or interior to any distribution of mass within any region having any of the geometries in nature,
2) Formulate all the nontrivial Einstein geometrical gravitational field equations and derive all their general solutions, and
3) Derive astrophysically most satisfactory unique solutions for application to the motions of all test particles and comparison with corresponding pure Newtonian results and applications. Therefore our goal in this research work has been completely achieved: to use the case of a spheroidal distribution of mass to show how the much vaunted Einstein's geometrical gravitational field equations may be solved exactly and analytically for any given distribution of mass within any region having any geometry.

On a final note, the theme studied in this research work is obviously very attractive as it is related to the expansion of our views to boundaries far away from our everyday experience, and opens beautiful horizons for possible laboratory, astrophysical and astronomical experiments. Naturally, Einstein's equations are of great importance to mankind, even if most people don't understand it clearly. By connecting the geometrical properties of space with the physical properties of matter, the equations regulate almost all of the space-time functions of our life. We are living in not just a mere three dimensional space, but in time that is manifested as the change of all physical structures (even the most stable physical structures change). The change of geometric formations changes the coordinate nets and hence, changes the geometrical structure of the space we observe. Einstein's equations rule this process. We are very optimistic that in the future, when people will fail to use oil as the source of energy, Einstein's equations will be the main engine for a theoretical physicist working on the sources of energy or related problems. People will turn their attention to more obvious and bizarre energetics than simply using oil or other fuels. As researchers in gravitational physics, we see many excellent sources of energy around us. These are the planets orbiting the Sun, rotating stars, stellar energy and many others. These sources are working from other principles than those known to modern theoretical physics. But these sources are not obvious as the self-rotating Sun (it should come to a halt after 2.5 revolutions due to internal viscosity) or the planets orbiting it (they also should experience a halt). The energy propelling these systems can be best understood from the space-time geometry and thus Einstein's theory of gravitation has a very promising future.

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