

Radial Distance on a Stationary Frame in a Homogeneous and Isotropic Universe

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Abstract: This paper presents a physical distance to all radial events in a homogeneous and isotropic universe as a transform from Friedman-Lemaître-Robertson-Walker (FLRW) coordinates, the model that solves the Einstein field equation for an ideal fluid. Any well behaved transform is also a solution. The problem is relating the coordinates of the transform to observables. In the present case the objective is to find T, R on a stationary frame that has the R be a physical observable for all distances. We do this by working backwards, assuming the form of the metric that we desire, with some undetermined coefficients. These coefficients are then related to the partial derivatives of the transform. The transformed coordinates T, R are found by the integration of partial differential equations in the FLRW variables. We show that dR has the same units as the radial differential of the FLRW metric, which makes it observable. We develop a criterion for how close the transformed T comes to an observable time. Close to the space origin at the present time, T also becomes physical, so that the stationary acceleration becomes Newtonian. We show that a galactic point on a R, T plot starts close to the space origin at the beginning, moves out to a physical distance and finite time where it can release light that will be seen at the origin at the present time. Lastly, because the observable R has a finite limit at a finite T for $t=0$ where the galactic velocity approaches the light speed, we speculate that the Universe filled with an ideal fluid as seen on clocks and rulers on the stationary frame has a finite extent like that of an expanding empty universe, beyond which are no galaxies and no space.

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§1. Introduction. The FLRW metric [1, 2] metric is derived for a homogeneous and isotropic universe. With the assumption of a stress-energy tensor for an ideal liquid, the solution of the Einstein Field Equation gives the FLRW model of the Universe [3–5] used in this paper. The FLRW variables (t, χ, θ, ϕ) are interpreted as co-moving coordinates with dt being a physical (observable) time, and $a(t)d\chi$ being a physical radial distance, and $a(t)$ being the scale factor of the FLRW universe. We visualize these galactic points representing galaxies as moving away from an observer (us) at the origin at a velocity measured by distance and time on our stationary frame. We can measure their velocity by the red shift of the lines of their spectrum, but their distance is more difficult. The proper distance $a(t)\chi$, the distance to χ at constant t , does not have a Minkowski metric with t (except for small χ) required for physical coordinates. Weinberg describes a sequence of physical distance measures, running from parallax measurements for nearby stars to standard candle measurements for more distant sources [5]. These all accurately represent the observables as derived from the FLRW coordi-

nates, but are not our stationary distance; they differ from each other and from proper distance for large red shifts.

This paper presents a physical distance on a stationary frame to all radial events in a homogeneous and isotropic universe as a transform from FLRW coordinates. Any well behaved transform from FLRW coordinates is also a solution of the field equation because tensor equations are invariant to transforms. The problem is relating the coordinates of the transform to observables. In the present case we would like the observables to be the time on clocks and the distance on rulers on the stationary frame attached to the spatial origin, but this is only possible close to the origin. The next best objective is to find T, R for all distances that become these observables close to the origin, and then have the R be a physical observable for all distances.

The key to showing that the distance is universal is to show that it is rigid with the same constant units as the FLRW radial differential distance on galactic points, which units are assumed to be physical. We outline here how we will do this. We will look for a transform to radial coordinates T, R from radial FLRW coordinates t, χ that are attached to the FLRW origin $\chi=0$. We can call T, R stationary because the velocity of the galactic point is zero at $\chi=0$.

To find the transform of t, χ to T, R we will start with a metric with dT, dR that is spherically symmetric in space, and find the coefficients of dT, dR by putting constraints on them and then integrating the resultant dT, dR , subject to the boundary conditions at $\chi=0$. The constraints we will impose is that the points of R be motionless with respect to each other (i.e., rigid), that the the line element ds of dT, dR be the same as FLRW, that its metric (the line element expressed in coordinate differentials) have no cross products $dT dR$ (i.e., diagonal), that the metric in dT, dR become Minkowski close to the origin, and that dR have the same units for all distances as the radial differential of the FLRW line element, $a(t)d\chi$. The resulting paths of galactic points and photons in the stationary coordinates closely resembles the known transform to T, R from t, χ for an expanding empty universe [8]. These have a Minkowski metric for all distances, and have universal physical coordinates T, R .

§2. Assumptions. We will use the word “line element” to represent the invariant ds and the word “metric” for the ds of a particular set of coordinates, and limit ourselves to the time and radial distance of spherical space symmetry (no transverse motion). The analysis depends on the following assumptions:

Assumption 1: The FLRW metric is a valid representation of a homogeneous and isotropic universe, whose coordinates (χ, θ, ϕ, t) are co-moving with the galactic points (representing smoothed out galaxies), and whose universe scale factor is $a(t)$.

Definition: “Physical” coordinates in time or distance over some interval will be defined as those having a linear relationship to the readings on a standard clock or a standard ruler, respectively (a particular case of observables). We call them physical because it describes coordinates on the rigid frame close to and attached to the point at the origin $\chi=0$.

In the limit of small intervals on an inertial frame, if physical time represents clocks at the location represented by the physical distance, according to General Relativity, their differentials will have a Minkowski metric. This definition includes the Schwarzschild coordinates of time and distance; but the Schwarzschild metric is not Minkowski because we represent time in its metric to be measured remotely from the distance. Note that $d\chi$ is not physical, but dt is, by our definition. Our objective is to find a *physical* distance for all radial events in the Universe.

Assumption 2: The radial coordinate differentials dt and $a(t)d\chi$ of FLRW are assumed physical.

This is plausible because they have the Minkowski metric in these dimensions and are assumed to be on adjacent inertial frames. dt will represent the time increment on galactic clocks that keep time like a standard clock, and $a(t)d\chi$ will represent the physical distance between adjacent galactic points separated by $d\chi$ as measured by light signals, i.e., a round trip light time would be $2a(t)\frac{d\chi}{c}$, the same method used to synchronize clocks on a rigid frame or to determine radar distance.

Definition: The coordinates $x^\mu(R, \theta, \phi, T)$ are transforms from FLRW with spherically symmetric space coordinates whose space origins are the same as FLRW, so $\chi=0$ and $V=0$ at $R=0$. We will therefore call them stationary coordinates.

Like all well behaved transforms, they satisfy the Einstein field equations, but we need to relate them to observables. The physical velocity with respect to a galactic point t, χ of a point on R will be $V = a(t)\left(\frac{\partial\chi}{\partial t}\right)_R$. For this radial point R of x^μ , we can find contravariant velocity vectors $U^\mu = \frac{dx^\mu}{ds}$ and acceleration vectors $A^\mu = \frac{DU^\mu}{ds}$ whose components will transform the same as dx^μ . R is rigid because the radial component of U^μ in stationary coordinates is $\left(\frac{\partial R}{\partial T}\right)_R \frac{dT}{ds} \equiv 0$, so the points of R are motionless with respect to each other.

Assumption 3: R will measure physical distance on the stationary frame if its differential is physical for all distances.

Without this assumption, R might represent a measurement whose increments would sometimes be smaller and sometimes larger than a standard ruler, and therefore not qualify for a physical distance measure on a stationary frame.

Assumption 4: The unit of dR is the same unit as that of the radial differential distance of the FLRW metric $a(t)d\chi$ if $\left(\frac{\partial R}{\partial \chi}\right)_t = \left(\frac{a \partial x}{\partial R}\right)_T$. This is the mathematical relation that shows that two observers are using the same distance units if each measures the other's unit to be the same as the other measures his [6].

Assumption 5: The known solution of the field equations of General Relativity for the FLRW universe gives $a(t)$.

Because of the properties of Riemann tensors, any transform from FLRW will also be a solution. This allows evaluation of T, R of an event for a given universe energy density.

§3. Procedure for finding stationary coordinates using the velocity V . We assume that the concentrated lumps of matter, like stars and galaxies, can be averaged to the extent that the universe matter can be considered continuous, and that the surroundings of every point in space can be assumed isotropic and the same for every point.

By embedding a maximally symmetric (i.e., isotropic and homogeneous) three dimensional sphere, with space dimensions r, θ , and ϕ , in a four dimension space which includes time t , one can obtain a differential line element ds [5, p. 403] such that

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (1)$$

where

$$r = \begin{cases} \sin \chi, & k = 1 \\ \chi, & k = 0 \\ \sinh \chi, & k = -1 \end{cases} \quad (2)$$

while k is a spatial curvature determinant to indicate a closed, flat, or open universe, respectively, and

$$d\chi^2 \equiv \frac{dr^2}{1 - kr^2}, \quad (3)$$

where $a(t)$ is the cosmic scale factor multiplying the three dimensional spatial sphere, and the differential radial distance is $a(t)d\chi$.

The resulting equation for the differential line element becomes the FLRW metric:

$$ds^2 = c^2 dt^2 - a(t)^2 [d\chi^2 + r^2 d\omega^2], \quad (4)$$

where $d\omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. For radial motions this metric becomes Minkowski in form with a differential of physical radius of $a(t) d\chi$.

We would now like to find radial transforms that will hold for all values of the FLRW coordinates and have a Minkowski metric for small distances from the origin as required if dT and dR are to be the time and distance on our stationary frame. The most general line element for a time-dependent spatially spherically symmetric (i.e., isotropic) line element [5] is

$$ds^2 = c^2 A^2 dT^2 - B^2 dR^2 - 2cCdTdR - F^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (5)$$

where A , B , C , and F are implicit functions of T and R , but explicit functions of t and χ .

We would like T and R to locate the same radial events as t and χ , so $T = T(t, \chi)$ and $R = R(t, \chi)$. Therefore

$$\left. \begin{aligned} dT &= T_t dt + T_\chi d\chi = \frac{1}{c} T_t d\hat{t} + T_\chi d\chi \\ dR &= R_t dt + R_\chi d\chi = \frac{1}{c} R_t d\hat{t} + R_\chi d\chi \end{aligned} \right\}, \quad (6)$$

where the subscripts indicate partial derivatives with respect to the subscript variable and $d\hat{t} = c dt$.

We will look for transformed coordinates which have their origins on the same galactic point as $\chi = 0$, so $R = 0$ when $\chi = 0$, where there will be no motion between them, and where T is t , since the time on clocks attached to every galactic point is t , including the origin. We will keep the same angular coordinates for the transform and make $F = ar$ to correspond to the FLRW metric, but will find only radial transforms where the angular differentials are zero.

Let us consider a radial point at R . When measured from the FLRW system, it will be moving at a velocity given by

$$V = a(t) \left(\frac{\partial \chi}{\partial t} \right)_R \equiv c \hat{V}, \quad (7)$$

This velocity will be the key variable that will enable us to obtain radial transforms of the full radial coordinates.

We will first use this to find the components in the FLRW coordi-

nates $(\chi, 0, 0, \hat{t})$ for the contravariant velocity vector U^μ of the point at R . To get the time component, we divide (4) by $d\hat{t}^2$ with $d\omega = 0$ to obtain

$$\left(\frac{ds}{d\hat{t}}\right)^2 = 1 - a(t)^2 \left(\frac{d\chi}{d\hat{t}}\right)^2 = (1 - \hat{V}^2) \equiv \frac{1}{\gamma^2}. \quad (8)$$

To get the spatial component, we use the chain rule applied to (7) and (8):

$$\frac{d\chi}{ds} = \frac{d\chi}{d\hat{t}} \frac{d\hat{t}}{ds} = \frac{\hat{V}}{a} \gamma. \quad (9)$$

Thus, the FLRW components of U^μ are

$$U^\mu = \left(\frac{\gamma \hat{V}}{a}, 0, 0, \gamma\right). \quad (10)$$

with the time component listed last.

The transformed coordinates are (R, θ, ϕ, T) . To get U^μ in these coordinates, we make the spatial components be zero as required for it to be a vector velocity of the point R (see the definitions under Assumption 3). This assumption means that a test particle attached to the radial coordinate will feel a force caused by the gravitational field, but will be constrained not to move relative to the coordinate. Alternatively, a co-located free particle at rest relative to the radial point will be accelerated by the same force, but will thereafter not stay co-located. This means that the radial point is not inertial, except when close to the origin.

The time component of the transformed vector is $\frac{dT}{ds} = \frac{1}{cA}$. This makes the vector U^μ in the transformed coordinates be

$$U^\mu = \left(0, 0, 0, \frac{1}{cA}\right). \quad (11)$$

Since U^μ is to be contravariant, its components will transform the same as dT, dR in (6):

$$\left. \begin{aligned} \frac{1}{cA} &= \frac{1}{c} T_t \gamma + \frac{1}{a} T_\chi \gamma \hat{V} \\ 0 &= \frac{1}{c} R_t \gamma + \frac{1}{a} R_\chi \gamma \hat{V} \end{aligned} \right\}. \quad (12)$$

Manipulating the second line of (12) gives

$$\hat{V} = -\frac{a R_t}{c R_\chi}. \quad (13)$$

If we invert (6), we get

$$\left. \begin{aligned} d\hat{t} &= \frac{1}{D} (R_\chi dT - T_\chi dR) \\ d\chi &= \frac{1}{D} \left(-\frac{1}{c} R_t dT + \frac{1}{c} T_t dR \right) \end{aligned} \right\}, \quad (14)$$

where

$$D = \frac{1}{c} T_t R_\chi - \frac{1}{c} R_t T_\chi = \frac{1}{c} T_t R_\chi \left(1 + \hat{V} \frac{cT_\chi}{aT_t} \right), \quad (15)$$

using (13).

We can enter $d\hat{t}$ and $d\chi$ of (14) into the FLRW metric (4). One way to make the line element ds to be the same as FLRW is to equate these coefficients individually with those of (5):

$$A^2 = \frac{1}{T_t^2} \left[\frac{1 - \hat{V}^2}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2} \right], \quad (16)$$

$$B^2 = \frac{a^2}{R_\chi^2} \left[\frac{1 - \left(\frac{cT_\chi}{aT_t}\right)^2}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2} \right], \quad (17)$$

$$C = -\frac{a}{T_t R_\chi} \left[\frac{\hat{V} + \frac{cT_\chi}{aT_t}}{\left(1 + \hat{V} \frac{cT_\chi}{aT_t}\right)^2} \right]. \quad (18)$$

If we put $ds=0$ in (5), we obtain a coordinate velocity of light v_p :

$$\frac{v_p}{c} = \left(\frac{\partial R}{c \partial T} \right)_s = -\frac{C}{B^2} \pm \sqrt{\left(\frac{C}{B^2} \right)^2 + \frac{A^2}{B^2}}. \quad (19)$$

The equations for A , B , and v_p simplify for a diagonal metric ($C=0$). From (18) we get

$$\frac{cT_\chi}{aT_t} = -\hat{V}. \quad (20)$$

So rigidity gives a relation of dR to V (13), and diagonality gives a relation of dT to V (20). Thence, equations (16), (17), and (19) become

$$A = \frac{\gamma}{T_t} = \frac{t_T}{\gamma}, \quad (21)$$

$$B = \frac{a\gamma}{R_\chi} = \frac{a\chi_R}{\gamma}, \quad (22)$$

$$\frac{v_p}{c} = \frac{A}{B}, \quad (23)$$

where we have used (14) with $C=0$ to obtain the inverse partials. This metric becomes the physical Minkowski (\hat{M}) when $A \rightarrow 1$, $B \rightarrow 1$, $C \rightarrow 0$ and $ar \rightarrow R$. Thus, the coordinate light speed for the stationary coordinates starts at c for $R=0$ where $T_t = \gamma = \frac{R_x}{a} = 1$, and increases by the ratio $\frac{A}{B}$ as R increases.

We can say something about the physicality of the coordinates with the use of criteria developed by Bernal et al. [6]. They developed a theory of fundamental units based on the postulate that two observers will be using the same units of measure when each measures the other's differential units at the same space-time point compared to their own and finds these reciprocal units to be equal. Thus, even if $A \neq 1$, dR will be physical if $C=0$ and $B=1$ because then $\frac{R_x}{a} = a\chi_R = \gamma$ (22), and dR is physical because it uses the same measure of distance as $ad\chi$, which we assume is physical. Then R uses physical units when dR is integrated indefinitely out to the visible horizon.

At this point we would like to examine quantitatively how far from the \hat{M} metric our stationary metric is allowed to be in order for its coordinates to reasonably represent physical measurements. We can consider the coefficients A , B , and C one at a time departing from their value in the \hat{M} metric. Thus, let us consider the physical distance case $B=1$, $C=0$ and examine the possible departure of the time rate in the transform from that physically measured. Then, from equations (21): $T_t = \frac{\gamma}{A}$, $t_T = \gamma A$. Thus, $1-A$ represents a fractional increase from γ in the transformed time rate T_t , and thus the fractional increase from the physical T_t of an inertial rod at rest with the stationary coordinates at that point. We can make a contour of constant A on our world map to give a limit for a desired physicality of the stationary time.

§4. Stationary physical distance using a diagonal metric. For diagonal coordinates with physical dR for all t and χ , $B=1$, so (17) becomes

$$R_\chi = a\gamma. \quad (24)$$

By integration we find

$$R = a \int_0^\chi \gamma \partial\chi_t, \quad (25)$$

where the subscript on $\partial\chi_t$ represents integration of χ at constant t . Partial differentiation with respect to t gives

$$R_t = c\dot{a} \int_0^\chi \gamma \partial\chi_t + a \int_0^\chi \gamma_t \partial\chi_t, \quad (26)$$

where the dot represents differentiation by cdt . We can then find \hat{V} from (13), (24), and (26) as

$$\hat{V} = -\frac{R_t}{c\gamma} = -\frac{1}{c\gamma} \left[c\dot{a} \int_0^\chi \gamma \partial_{\chi t} + a \int_0^\chi \gamma_t \partial_{\chi t} \right]. \quad (27)$$

This is an integral equation for \hat{V} . It can be converted into a partial differential equation by multiplying both sides by γ and partial differentiating by χ :

$$\gamma^2 \left[\hat{V}_x + \frac{a}{c} \hat{V} \hat{V}_t \right] = -\dot{a} = -\frac{1}{c} \frac{da}{dt}. \quad (28)$$

General solutions of (28) for $\hat{V}(t, \chi)$ are found in the Appendix. $T(t, \chi)$, and $R(t, \chi)$ are also found there by their dependence on $\hat{V}(t, \chi)$. This will complete our search for a transform from FLRW that will satisfy the General Relativity field equation with a physical R to all radial events.

§5. Interpretation. We can use these solutions to show on a R, T plot the paths of galaxies ($\chi = \text{constant}$) and photons ($ds = 0$). We will use Ω as defined by Peebles [3] (see Appendix D). Thus $\Omega = 1$ is a flat universe ($k = 0$) with no cosmological constant ($\Lambda = 0$). These are shown in Figs. 1 and 2. An approximate upper limit of physicality is shown by the dashed line for $A = 1.05$. Below this line, T is close to physical. R vs. T at $t = 0$ provides a horizon, where the visible universe has a finite physical distance, but where T is non-physical.

Fig. 1 coordinates are both physical, the physical distance R to a galactic point at χ characterized by the red shift $z = -1 + \left(\frac{t_0}{t}\right)^{2/3}$ and the time at the origin (or on any galactic point) $\frac{t}{t_0}$. The light we see now at the origin originates from a galactic point when its path crosses the light path, where the physical distance is as shown. Notice that the light comes monotonically towards us even from the farthest galactic point, but its coordinate speed slows down the farther it is away in these coordinates where the clocks measuring t are on moving galactic points whereas the rulers measuring R are on the stationary frame. Thus, $\left(\frac{a\partial\chi}{\partial t}\right)_s = c$ when clocks and rulers are on the same frame, but $\left(\frac{\partial R}{\partial t}\right)_s$ is a coordinate light speed not equal to c (except close to the origin where t approaches T) because R and t are on different frames. This is much like the coordinate light speed of the Schwarzschild coordinates [5] where clocks and distances are at different locations.

Although the distance to the origin in Fig. 2 is physical, the co-located time coordinate T is not, but comes close to it near the space

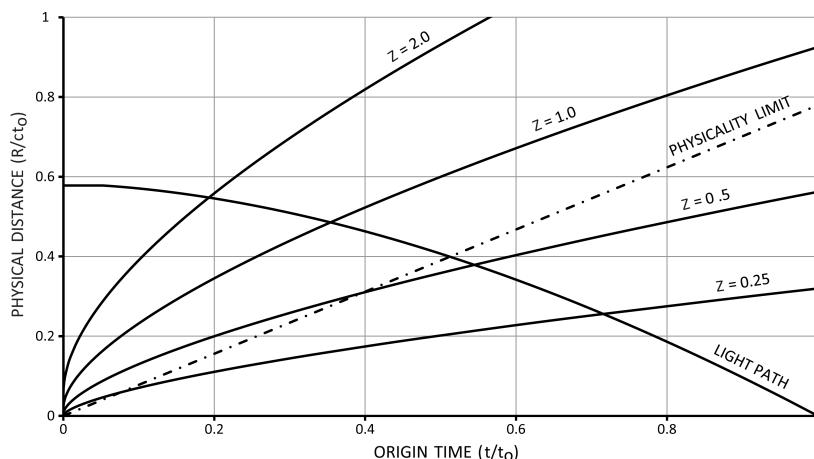


Fig. 1: Physical distance R/c_0t_0 for a flat universe ($\Omega = 1$) vs. physical time at the origin t/t_0 (or galactic time). The paths of galactic points representing galaxies are characterized by their red shift z . The transformed time (not shown) is close to physical only below the PHYSICALITY LIMIT line where $A = 1.05$. Light comes towards the origin along the LIGHT PATH from all radial points of the universe, travelling slower than its present speed in these coordinates where the clocks are on different frames from the rulers (like the Schwarzschild coordinates). The galactic paths show the expanding universe in physical coordinates, some travelling faster than the light speed in these coordinates.

origin. For distances and times below the physicality line $A = 1.05$, T_t is less than the physical time by only 5% on a co-located stationary inertial frame.

As such, the physical interpretation seems very clear. A galactic point seems to start at $R = 0$, $T = 0$ and moves out to the time it emits its light that can be later seen at the origin at $T = t_0$. The farthest galactic points travel out the fastest and release their light the earliest, but even for the most remote galactic point, the distance never gets greater than $0.58 c_0t_0$, which can be considered to be the size of the visible universe when the farthest light was emitted.

Near the horizon, some oddities occur in this interpretation; when $t \rightarrow 0$, T comes close but is not zero for most galactic paths, but also is not physical at that point. R uses physical rulers at that point, but also is not quite zero. This discrepancy becomes the largest for the farthest galactic points. The lack of physicality near the horizon is presumably what causes the transform to show non-zero times and distances at $t = 0$,

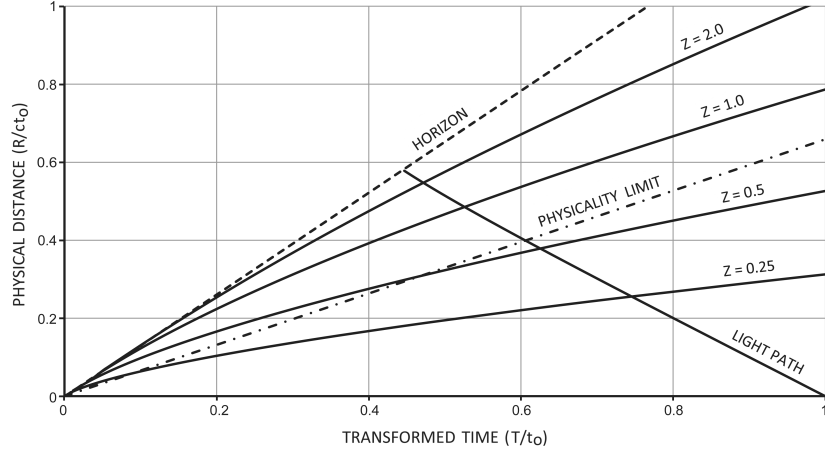


Fig. 2: Physical distance ($R/c_0 t_0$) for a flat universe ($\Omega = 1$) vs. the stationary local time T/t_0 , which is close to physical below the PHYSICALITY LIMIT line ($A = 1.05$). A galactic point starts close to the physical origin at $T = 0$ and travels out to where it may emit light when it crosses the LIGHT PATH that will travel close to the physical light speed and be visible later at the physical origin at $T = t_0$. The line labelled HORIZON represents singularities ($t = 0$ and $\gamma = \infty$), beyond which there are no galactic points, and no space.

$\gamma = \infty$ and the farthest galactic points to go close to the stationary coordinate light speed which approaches $1.3c$ near the horizon. Of course, if there were to be a different stationary transform with constant inertial light speed that had a rigid physical distance, it might have a different universe radius, but this seems unlikely. (A possible alternative with different assumptions is explored in [7].)

The above interpretation is strengthened by examining the physical coordinates of an expanding empty FLRW universe, a solution that was first published by Robertson with $a = ct$ [8]:

$$R = ct \sinh \chi, \quad T = t \cosh \chi. \quad (29)$$

Robertson showed that these transformed coordinates obeyed the Minkowski metric for all t, χ , as is required of physical coordinates in empty space. This solution is plotted in Fig. 3 with the same format as Fig. 2. The paths are very similar except that all the paths are straight with no universe density to curve them, and the galactic paths do not have a gap at the singularity at $t = 0$. Notice that along the light path that $\chi_p = \ln(\frac{t_0}{t})$, and $R_p = \frac{1}{2} ct_0 \left(1 - \frac{t^2}{t_0^2}\right)$ so that at $t = 0$, the extent of the

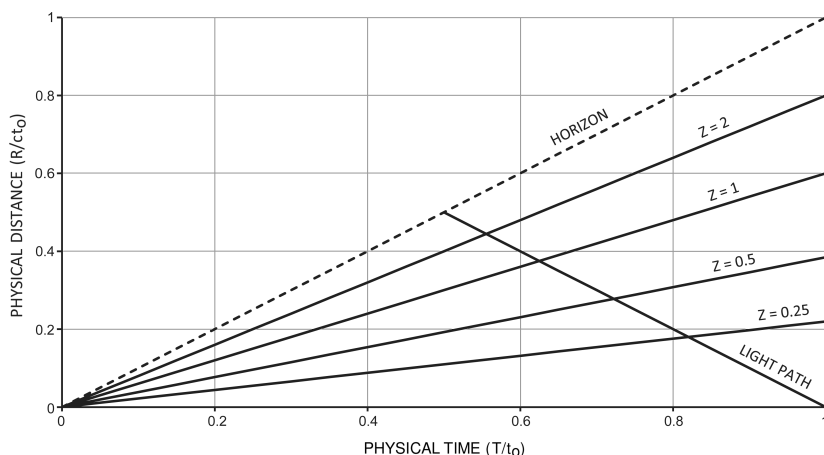


Fig. 3: Physical distance (R/ct_0) for the empty expanding universe ($\Omega = 0$, $\Omega_r = 1$) plotted against the transformed time (T/t_0) on clocks attached at R for various galaxy paths (labelled by their red shift z) and for the light path that arrives at the origin at $T = t_0$. The horizon is the locus of points where $t = 0$. All lines are straight and physical, since there is no space curvature. The remotest galactic point travels from the origin at $T = 0$ out to $\frac{1}{2} ct_0$ at the light speed c . There is no space outside the horizon to contain any galactic points.

visible universe is $R_p = \frac{1}{2} ct_0$. In terms of the clocks at the origin that read t , light seems to come monotonically towards us from this distance, the coordinate light speed $\left(\frac{\partial R}{\partial t}\right)_s$ slowing to zero at $t = 0$.

It's natural to wonder what is outside the horizon. In this empty universe, all galactic points are within the horizon. Outside the horizon not only are there no galactic points, there is *no space* when viewed from a stationary frame. Like a Fitzgerald contraction all differential radial distances shrink to zero at the horizon as the galactic points approach the light speed. From one of those points, it would have a different stationary frame and so would see a finite space in its vicinity and would see a different horizon.

For a finite universe energy density, many higher values of χ are not included inside the horizon. All of this is clouded by the non-physicality of T close to the horizon. But it would appear that all galactic matter starts outward from $R = 0$ at $T = 0$ travelling close to the light speed near the horizon, the largest χ travelling the fastest. As each χ_1 is slowed down by the inward gravitational pull of the mass of galaxies inside it, it departs from the horizon after a time $T_1(\chi_1)$ at a distance

$R_1(\chi_1)$, where it can emit light that will be the earliest light visible at the origin at time $T = t_1$, so $\chi_1 = 3\left(\frac{t_1}{t_0}\right)^{1/3}$ for $\Omega = 1$, see (54). Those with a $\chi > \chi_1$ will not be visible at the origin at $T = t_1$, because they must be inside the horizon to emit light, presumably because of the singularities in γ and in energy density along the horizon. Thus, only a limited number of galactic points can be visible at the origin at the present time, those with $t_1 \leq t_0$.

This is a different picture from a universe that is full of galactic points beyond a visible horizon, but are invisible only because there is not enough time to see them. I am suggesting that like the empty universe, there are no galactic points and no space beyond the horizon when viewed from a stationary frame.

§6. Newtonian gravity for flat space ($\Omega = 1$) close to the origin. In Appendix C, the acceleration vector A^μ of a point on R is found in FLRW coordinates and in stationary coordinates. The latter is solved for a flat universe ($k = 0$) with no dark energy $\Omega = 1$ in normalized coordinates ($\frac{\chi a_0}{ct_0} \rightarrow x$, $\frac{t}{t_0} \rightarrow t$, see Appendix C.2). For small $u = \frac{x}{t^{1/3}}$, the gravitational acceleration g goes to zero as $-\frac{2u}{9t} = -\frac{2R}{9}$. Since small u is the region with physical coordinates, it is interesting to express g and R in unnormalized coordinates:

$$-g \longrightarrow \frac{2}{9} \frac{c}{t_0} \frac{R}{ct_0} = \frac{4\pi\rho_0 GR}{3} = \frac{GM_0}{R^2}, \quad (30)$$

where we have used $\rho = \rho_0\left(\frac{a_0}{a}\right)^3$ in (39) and M_0 as the present universe mass inside the radius R . Thus, the gravitational force in the stationary coordinates used here near the origin is the same as Newtonian gravity, as one might expect if both T and R are physical near the origin and if $V \ll c$.

§7. Conclusion. We have found a transform to stationary coordinates from the FLRW coordinates for radial events that has a physical radial distance to each event. Although the time of the transform is not physical for all events, it is physical close to the space origin. When the physical radius is plotted against the physical time at the origin, the coordinate light speed $\frac{dR}{dt}$ slows down the farther out in the Universe it is observed, much like the coordinate light speed of Schwarzschild coordinates. But, since the time and distance of the stationary coordinates are both local on the same frame, the light speed in these coordinates remains fairly constant out to the far reaches of the Universe. Near the origin in these coordinates, the gravitational acceleration becomes New-

tonian for a flat universe ($\Omega=1$). The stationary coordinates indicate that galactic points originate at $T=0$, move out to a finite distance in a finite time when it releases light that we can see at the present time. For the most distant galactic points the distance at emission is about half of the so-called time-of-flight distance $c(t_0-t_e)$. An expanding horizon occurs in the stationary coordinates where the radial velocity of the galactic points approaches the light speed, outside of which there are no galactic points, and no space. At the time of the release of the earliest light that we could see today, the horizon was at $R=0.58ct_0$ in the stationary coordinates for a flat universe ($\Omega=1$) with no dark energy.

Appendix A. Stationary coordinates for any $a(t)$

A.1. Solution for \hat{V} . Equation (28) can be solved as a standard initial-value problem. Let $W \equiv -\hat{V}$. Then (28) becomes

$$W_\chi - \frac{a}{c} WW_t = \frac{1}{c} \frac{da}{dt} (1 - W^2). \quad (31)$$

Define a characteristic for $W(t, \chi)$ by

$$\left(\frac{\partial t}{\partial \chi} \right)_c = -\frac{a}{c} W \quad (32)$$

so

$$\left(\frac{\partial W}{\partial \chi} \right)_c = \frac{1}{c} \frac{da}{dt} (1 - W^2) \quad (33)$$

(the subscript c here indicates differentiation along the characteristic). If we divide (33) by (32) we get

$$\left(\frac{\partial W}{\partial t} \right)_c = -\frac{1}{a} \frac{da}{dt} \frac{(1 - W^2)}{W}. \quad (34)$$

This can be rearranged to give

$$\frac{W(\partial W)_c}{W^2 - 1} = \frac{(\partial a)_c}{a}. \quad (35)$$

This can be integrated along the characteristic with the boundary condition at $\chi = 0$ that $W=0$ and $a = a_c$:

$$1 - W^2 = \frac{a^2}{a_c^2} = \frac{1}{\gamma^2}. \quad (36)$$

This value for W (assumed positive for expanding universe) can be inserted into (32) to give

$$\left(\frac{\partial t}{\partial \chi}\right)_c = -\frac{a}{c} \sqrt{1 - \frac{a^2}{a_c^2}}. \quad (37)$$

We can convert this to a differential equation for a by noting that $c(\partial t)_c = (\partial \hat{t})_c = \frac{1}{\hat{a}}(\partial a)_c$

$$\left(\frac{\partial a}{\partial \chi}\right)_c = -a \hat{a} \sqrt{1 - \frac{a^2}{a_c^2}}. \quad (38)$$

The \hat{a} can be found as a function of a from the well known solution of the General Relativity field equation for the FLRW universe [3]:

$$\dot{a} = a \sqrt{\frac{8\pi G}{3c^2} \rho - \frac{k}{a^2} + \frac{\Lambda}{3}}, \quad (39)$$

where ρ is the rest mass density of the ideal liquid assumed for the universe that can be found as a function of a (see Appendix D); G is the gravitation constant; k is the constant in the FLRW metric (2); and Λ is the cosmological constant possibly representing the dark energy of the universe.

Equation (38) can be integrated along the characteristic with constant α_c , starting with $\alpha = \alpha_c$ at $\chi = 0$. This will give $\chi = X(\alpha, \alpha_c)$. This can be inverted to obtain $\alpha_c(\alpha, \chi)$. When this is inserted into (36), we have a solution to (31) for $W(\alpha, \chi)$. Integration of (39) gives $a(t)$ and thus W as a function of t, χ .

A.2. Obtaining T, R from \hat{V} . Equations (7), (13), and (20) show that

$$W = -\frac{a}{c} \left(\frac{\partial \chi}{\partial t}\right)_R = \frac{a}{c} \frac{R_t}{R_\chi} = \frac{c}{a} \frac{T_\chi}{T_t} \quad (40)$$

so

$$T_\chi - \frac{a}{c} W T_t = 0. \quad (41)$$

Thus T has the same characteristic as W (32), so that $\left(\frac{\partial T}{\partial \chi}\right)_c = 0$, and T is constant along this characteristic:

$$T(t, \chi) = T(t_c, 0) = t_c \equiv t[\alpha_c(t, \chi)], \quad (42)$$

where $t(\alpha)$ is given by the integration of (39) and $\alpha_c[\alpha(t), \chi]$ is found by inverting the integration of (39). This gives us the solution for $T(t, \chi)$

and A

$$A = \frac{\gamma}{T_t} = \frac{a_c}{a} \left(\frac{\partial t}{\partial t_c} \right)_\chi = \frac{a_c}{a} \frac{\frac{da_c}{dt_c}}{\frac{da}{dt}} \left(\frac{\partial a}{\partial a_c} \right)_\chi. \quad (43)$$

The solution for R can be obtained by integrating (25), using γ from (36) and $a_c(t, \chi)$ from (39):

$$R(t, \chi) = a \int_0^\chi \gamma \partial \eta_t = \int_0^\chi a_c(t, \eta) \partial \eta_t. \quad (44)$$

Alternatively, for ease of numerical integration we would like to integrate dR along the same characteristic as T and W . This can be obtained from the partial differential equation

$$\left(\frac{\partial R}{\partial \chi} \right)_c = R_\chi + R_t \left(\frac{\partial t}{\partial \chi} \right)_c. \quad (45)$$

If we insert into equation (45) the values for these three quantities from (24), (40), and (32), we get

$$\left(\frac{\partial R}{\partial \chi} \right)_c = \gamma a + \frac{cW}{a} \gamma a \left(-\frac{cW}{a} \right) = \frac{a}{\gamma} = \frac{a^2}{a_c}. \quad (46)$$

Appendix B. Similarity solutions for flat universe ($\Omega = 1$). But I have found a simpler integration of (28) for the special case of $\Omega = 1$ where the General Relativity solution is $a = a_0 \left(\frac{t}{t_0} \right)^{2/3}$ [3]. To simplify notation, let us normalize: $\frac{t}{t_0} \rightarrow t$, $\frac{a}{a_0} = \alpha = t^{2/3}$, and $\frac{\chi a_0}{ct_0} \rightarrow x$, $\frac{T}{t_0} \rightarrow T$, $\frac{R}{ct_0} \rightarrow R$, and let $W = -\hat{V}$.

B.1. Ordinary differential equation for \hat{V} or $-W$. Equation (28) then becomes

$$W_x - t^{2/3} W W_t = \frac{2}{3} t^{-1/3} (1 - W^2). \quad (47)$$

This can be converted into an ordinary differential equation by letting

$$u \equiv \frac{x}{t^{1/3}} \quad (48)$$

so that (47) becomes

$$W' \left(1 + \frac{uW}{3} \right) = \frac{2}{3} (1 - W^2), \quad (49)$$

where the prime denotes differentiation by u .

B.2. Ordinary differential equation for T and R . Similarly we can find ordinary differential equations for T and R by defining:

$$\frac{T}{t} \equiv q(u), \quad (50)$$

and

$$\frac{R}{t} \equiv s(u), \quad (51)$$

where $q(u)$ and $s(u)$, from (20) and (13), are given by the coupled ordinary differential equations:

$$q' \left(1 + \frac{uW}{3} \right) = qW, \quad (52)$$

and

$$s \left(W + \frac{u}{3} \right) = s. \quad (53)$$

It is useful to find that $q = \gamma^{3/2}$, $s = \gamma \frac{u+3W}{3}$, $s' = \gamma$, $A = \frac{\gamma}{T_t} = \frac{1}{\gamma} \left(1 + \frac{uW}{3} \right) = \frac{v_p}{c}$; so $T = t\gamma^{3/2}$, and $R = \frac{1}{3} t\gamma(u+3W)$.

For small values of u , $W = \frac{2u}{3}$, $q = 1 + \frac{u^2}{3}$, $s = u$, $\frac{v_p}{c} = 1 + O(W^4)$, and $R = t^{2/3}x = ax$. The light speed v_p measured on T, R remains close to c out to large R . We also note that $T_t \rightarrow 1 + \frac{W^2}{4}$, which is slower than a Lorentz requirement of $T_t \rightarrow \gamma \rightarrow 1 + \frac{W^2}{2}$.

As $t \rightarrow 0$, $u \rightarrow \infty$, $\gamma \rightarrow \kappa u^2$, $W \rightarrow 1 - \frac{1}{2\kappa^2 u^4}$, $q \rightarrow \kappa^{3/2} u^3$, and $s \rightarrow \frac{\kappa u^3}{3}$. The quantities T and R both remain finite at this limit with $T \rightarrow \kappa^{3/2} x^3$, and $R \rightarrow \frac{\kappa x^3}{3}$ with $\frac{T}{R} \rightarrow 3\kappa^{1/2}$. The κ is difficult to determine from the numerical integration because of the singularity at large u , but my integrater gives $\kappa = 0.0646$. The fact that T does not go to zero when t goes to zero results from equating T with t at $t=1$ and not at $t=0$.

The distance R and time T can be found from the numerical integration of the coupled ordinary differential equations. The paths of galactic points are those for constant x . The path photons have taken reaching the origin at t_1 is found by calculating x_p vs. t and using the transform to T, R . Thus, for $\Omega = 1$,

$$x_p = \int_{t_1}^t \frac{c}{a} dt = 3(t_1^{1/3} - t^{1/3}). \quad (54)$$

For light arriving now, $t_1 = 1$, the value of u_p becomes

$$u_p = 3 \left(\frac{1}{t^{1/3}} - 1 \right). \quad (55)$$

Note that $x_p \rightarrow 3$ as $t \rightarrow 0$ at the beginning of the path for photons. This makes the horizon at the earliest time of release of light visible today be $0.581ct_0$.

Appendix C. Gravitational field

C.1. Gravitational field in the FLRW and stationary coordinates. We wish to find the components of the radial acceleration of a test particle located at R in the stationary system. We will do this by calculating the FLRW components of the acceleration vector and find the transformed components by using the known diagonal transforms. For the FLRW components, we will use the metric

$$ds^2 = dt^2 - a^2 d\chi^2 - a^2 r^2 d\theta^2 - a^2 r^2 \sin^2 \theta d\phi^2. \quad (56)$$

Let

$$x^1 = \chi, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = \hat{t} = ct, \quad (57)$$

and the corresponding metric coefficients become

$$g_{44} = 1, \quad g_{11} = -a^2, \quad g_{22} = -a^2 r^2, \quad g_{33} = -a^2 r^2 \sin^2 \theta. \quad (58)$$

For any metric, the acceleration vector for a test particle is

$$A^\lambda = \frac{dU^\lambda}{ds} + \Gamma_{\mu\nu}^\lambda U^\mu U^\nu, \quad (59)$$

where the Γ 's are the affine connections and U^λ is the velocity vector of the test particle. In our case the test particle is at the point R on the transformed coordinate, but not attached to the frame so that it can acquire an acceleration. Instantaneously, it will have the same velocity as the point on the transformed coordinate, and its velocity and acceleration vectors will therefore transform the same as the point (12).

We will be considering accelerations only in the radial direction so that we need find affine connections only for indices 1, 4. The only non-zero partial derivative with these indices is

$$\frac{\partial g_{11}}{\partial x^4} = -2a\dot{a}. \quad (60)$$

The general expression for an affine connection for a diagonal metric is

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2g_{\lambda\lambda}} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right). \quad (61)$$

The only three non-zero affine connections with 1, 4 indices are

$$\Gamma_{11}^4 = a\dot{a}, \quad \Gamma_{41}^1 = \Gamma_{14}^1 = \frac{\dot{a}}{a}. \quad (62)$$

The acceleration vector in FLRW coordinates of our test particle moving at the same velocity as a point on the transformed frame becomes

$$\left. \begin{aligned} A^{\hat{t}} &= \frac{dU^4}{ds} + \Gamma_{11}^4 U^1 U^1 \\ A^x &= \frac{dU^1}{ds} + \Gamma_{41}^1 (U^4 U^1 + U^1 U^4) \end{aligned} \right\}. \quad (63)$$

Using U^4 and U^1 in (10), we find

$$\left. \begin{aligned} A^{\hat{t}} &= \gamma \left(\frac{\partial \gamma}{\partial \hat{t}} \right)_R + a \dot{a} \frac{\gamma^2 \hat{V}^2}{a^2} = \gamma^4 \hat{V} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \gamma^2 \hat{V}^2 \\ A^x &= \gamma \left[\frac{\partial}{\partial \hat{t}} \left(\frac{\gamma \hat{V}}{a} \right) \right]_R + 2 \frac{\dot{a}}{a} \frac{\gamma^2 \hat{V}}{a} = \frac{\gamma^4}{a} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a^2} \gamma^2 \hat{V} \end{aligned} \right\}. \quad (64)$$

Since the acceleration vector of the test particle at R in the stationary coordinates will be orthogonal to the velocity vector, it becomes

$$\left. \begin{aligned} A^T &= 0 \\ A^R &= \frac{DU^R}{ds} \equiv -\frac{g}{c^2} \end{aligned} \right\}. \quad (65)$$

Here A^R is the acceleration of a point on the R axis, so the gravitational field affecting objects like the galactic points is the negative of this. g is defined so that mg is the force acting on an object whose mass is m . Close to the origin, $g = \frac{d^2 R}{dT^2}$, the normal acceleration. Since the vector A^λ will transform like dT, dR (6):

$$A^R = \frac{1}{c} R_t A^{\hat{t}} + R_x A^x \quad (66)$$

so that

$$-\frac{g}{c^2} = \left[\gamma^4 \hat{V} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \gamma^2 \hat{V}^2 \right] \frac{1}{c} R_t + \left[\frac{\gamma^4}{a} \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a^2} \gamma^2 \hat{V} \right] R_x. \quad (67)$$

With the use of (13), this can be simplified to

$$-\frac{g}{c^2} = \frac{R_x}{a} \left[\gamma^2 \left(\frac{\partial \hat{V}}{\partial \hat{t}} \right)_R + \frac{\dot{a}}{a} \hat{V} \right]. \quad (68)$$

The acceleration g can be thought of as the gravitational field caused by the mass of the surrounding galactic points, which balances to zero at

the origin, where the frame is inertial, but goes to infinity at the horizon. It is the field which is slowing down the galactic points (for $\Lambda = 0$).

C.2. Gravitational force for flat universe ($\Omega = 1$). If we insert the values of V , R , and $\frac{a}{c}$ in Appendix C, (68), we obtain

$$g = -\frac{s'}{t} \left[\gamma^2 W' \left(\frac{u}{3} + W \right) - \frac{2W}{3} \right] \quad (69)$$

where g has the units $\frac{c\alpha}{t_0}$. The insertion of W' and s' into (69) gives

$$-g = \frac{2}{9\gamma t} \frac{u}{1 + \frac{uW}{3}}. \quad (70)$$

Appendix D. Gravitational field Peebles has shown a convenient way to represent equation 39 (see [3, p.312]), the solution of the field equation for the FLRW metric in a universe filled with an ideal liquid. He defines

$$\Omega \equiv \rho_0 \frac{8\pi G}{3c^2 H_0^2}, \quad (71)$$

and

$$\Omega_r \equiv \frac{-k}{H_0^2 a_0^2}, \quad (72)$$

and

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}. \quad (73)$$

Here ρ_0 , H_0 , and a_0 are the energy density, Hubble constant, and universe scale factor, respectively, at the present time t_0 . For very small a there will also be radiation energy term $\Omega_R \approx 2 \times 10^{-5}$ [3].

Let $\alpha = \frac{a}{a_0}$. It is found by the solution of the ordinary differential equation:

$$\frac{1}{\alpha} \frac{d\alpha}{dt} \equiv H = H_0 E(\alpha), \quad (74)$$

where the normalized Hubble ratio E is

$$E(\alpha) = \sqrt{\frac{\Omega_R}{\alpha^4} + \frac{\Omega}{\alpha^3} + \frac{\Omega_r}{\alpha^2} + \Omega_\Lambda}. \quad (75)$$

The Ω s are defined so that

$$\Omega_R + \Omega + \Omega_r + \Omega_\Lambda = 1. \quad (76)$$

At $t = t_0$: $\alpha = 1$ and $E = 1$.

The cosmic time t measured from the beginning of the FLRW universe becomes

$$cH_0 t = \int_0^\alpha \frac{1}{\alpha E} d\alpha. \quad (77)$$

For a flat universe with $\Omega = 1$ and $\Omega_R = \Omega_r = \Omega_\Lambda = 0$:

$$\alpha = \left(\frac{t}{t_0} \right)^{2/3}, \quad t_0 = \frac{2}{3cH_0}. \quad (78)$$

Acknowledgements. I wish to acknowledge the invaluable help given by Paul Fife, University of Utah Mathematics Department.

Submitted on April 29, 2011

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Vol. 4, 2011

ISSN 1654-9163

— THE —

ABRAHAM ZELMANOV JOURNAL

The journal for General Relativity,
gravitation and cosmology

TIDSKRIFTEN —

ABRAHAM ZELMANOV

Den tidskrift för allmänna relativitetsteorin,
gravitation och kosmologi

Editor (redaktör): Dmitri Rabounski
Secretary (sekreterare): Indranu Suhendro

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