A Three-Dimensional Charged Black Hole Inspired by Non-Commutative Geometry

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Abstract: We find a new charged black hole solution in three-dimensional anti-de Sitter (AdS) space using an anisotropic perfect fluid inspired by a non-commutative black hole as the source of matter and a Gaussian distribution of electric charge. We deduce the thermodynamical quantities of this black hole and compare them with those of a charged Banados-Teitelboim-Zanelli (BTZ) solution.

Contents:

§1. Introduction. The theoretical discovery of radiating black holes [1] disclosed the first physically relevant window regarding the mysteries of quantum gravity. The string-black hole correspondence principle [2] suggests that in this extreme regime stringy effects cannot be neglected. One of the most interesting outcomes of string theory is that target space-time coordinates become non-commuting operators on a D-brane [3]. Thus, string-brane coupling has put in evidence the necessity of space-time quantization. Recently, an improved version of field theory of a non-commutative space-time manifold has been proposed as a cheaper way to reproduce the string phenomenology, at least in the low-energy limit. In this proposal, non-commutativity is encoded in the commutator

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix which determines the fundamental cellular discretization of space-time much in the same way as the Planck constant $\hbar$ discretizes the phase space. This proposal provides a black hole with a minimum scale $\sqrt{\theta}$ known as the non-commutative black hole [4–8], whose commutative limit is the Schwarzschild metric. The thermodynamics and evaporation process of the non-commutative black hole has been studied in [9], while the entropy issue is discussed in

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[10, 11] and the Hawking radiation in [12]. Charged non-commutative black holes have been studied in [13,14] and recently, a non-commutative three-dimensional black hole whose commutative limit is revealed by the non-rotating Banados-Teitelboim-Zanelli (BTZ) solution was studied in [15], while the three-dimensional rotating counterpart of it was deduced in [16].

In this paper, we construct a new charged black hole in AdS_3 spacetime using an anisotropic perfect fluid inspired by the four-dimensional non-commutative black hole as the source of matter while considering a Gaussian distribution of electric charge. The resulting solution exhibits two horizons that degenerate into one in the extremal case. We compare the thermodynamics of this non-commutaive black hole with that of the charged BTZ solution [17, 18].

§2. Derivation of the charged solution in three dimensions. In the analysis of black holes in the framework of non-commutative spaces, one has to solve the corresponding field equations. As argued in [6,19] it is not necessary to change the Einstein tensor part of the field equations because the non-commutative effects act only on the matter source. The underlying philosophy of this approach is to modify the distribution of point-like sources in favour of smeared objects. This is in agreement with the conventional procedure for the regularization of ultra-violet divergences by introducing a cut-off. As a gravitational analogue of the non-commutative modification of quantum field theory [4], we conclude that in General Relativity, the effect of non-commutativity can be taken into account by keeping the standard form of the Einstein tensor on the left-hand side of the field equations as well as by introducing a modified energy-momentum tensor as a source on the right-hand side.

Therefore, one way of implementing the effect of smearing is the following substitution rule: in three dimensions, the Dirac delta function $\delta^{3\mathrm{D}}(r)$ is replaced by a Gaussian distribution with minimal width $\sqrt{\theta}$,

$$\rho\left(r\right) = \frac{M}{4\pi\theta}\, e^{-r^2/4\theta} \tag{2}$$

giving a mass distribution in the form

$$m\left(r\right) = 2\pi \int_0^r r' \rho\left(r'\right) dr' = M\left(1 - e^{-r^2/4\theta}\right). \tag{3}$$

As coordinate non-commutativity is a property of the space-time fabric itself, and not of its material content, the same smearing effect is expected to operate on electric charge [13,14]. Thus, a point-charge Q is spread throughout a minimal-width Gaussian charge cloud according to

$$\rho_e(r) = \frac{Q}{4\pi\theta}\, e^{-r^2/4\theta}. \tag{4}$$

For a static, circularly symmetric charge distribution, the current density J_μ is non-vanishing only along the time direction, i.e.

$$J^\mu = (\rho_e,\, 0,\, 0)\,. \tag{5}$$

In order to find a black hole solution in AdS_3 space-time, we recall the Einstein-Maxwell equations,

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi \left(T^{\text{matter}}_{\mu\nu} + T^{\text{electr}}_{\mu\nu}\right) + \frac{1}{\ell^2}\, g_{\mu\nu}\,, \tag{6}$$

$$\frac{1}{\sqrt{|g|}}\, \partial_\mu \big(\sqrt{|g|}\, F^{\mu\nu}\big) = J^\nu, \tag{7}$$

where ℓ is related with the cosmological constant by

$$\Lambda = -\frac{1}{\ell^2}\,. \tag{8}$$

The energy-momentum tensor for matter will take the anisotropic form

$$(T^\mu_\nu)^{\text{matter}} = \text{diag}\,(-\rho,\ p_r,\ p_\perp)\,. \tag{9}$$

In order to complete define this tensor, we rely on the covariant conservation condition $T^{\mu\nu}{}_{,\nu} = 0$. This gives the source as an anisotropic fluid of density ρ, radial pressure

$$p_r = -\rho \tag{10}$$

and tangential pressure

$$p_\perp = -\rho - r\,\partial_r \rho. \tag{11}$$

The electromagnetic energy-momentum tensor $T^{\text{electr}}_{\mu\nu}$ is defined in terms of $F_{\mu\nu}$ as

$$T^{\text{electr}}_{\mu\nu} = -\frac{2}{\pi}\left(F_{\mu\sigma} F^{\sigma\cdot}_{\cdot\nu} - \frac{1}{4}\, g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right). \tag{12}$$

By solving the Maxwell equations (7) with source (5), we obtain the electric field

$$E\,(r) = \frac{1}{r}\int_0^r r' \rho_e(r')\, dr' = \frac{Q}{2\pi r}\left(1 - e^{-r^2/4\theta}\right). \tag{13}$$

Using the static, circularly symmetric line-element

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2 d\varphi^2, \tag{14}$$

the Einstein field equations (6) are written accordingly as

$$\frac{1}{r}\frac{df}{dr} = -16\pi\rho - \frac{1}{2}\,E^2 + \frac{2}{\ell^2}\,, \tag{15}$$

$$\frac{d^2 f}{dr^2} = 16\pi\rho_\perp + \frac{1}{2}\,E^2 + \frac{2}{\ell^2}\,. \tag{16}$$

Solving the above equations, we find

$$f\,(r) = -8M\left(1 - e^{-r^2/4\theta}\right) + \frac{r^2}{\ell^2} - \\ - \frac{Q^2}{8\pi^2}\left[\,\ln|r| + \frac{1}{2}\,\mathrm{Ei}\Big(-\frac{r^2}{2\theta}\Big) - \mathrm{Ei}\Big(-\frac{r^2}{4\theta}\Big)\right], \tag{17}$$

where $\mathrm{Ei}\,(z)$ represents the exponential integral function,

$$\mathrm{Ei}\,(z) = -\int_{-z}^{\infty}\frac{e^{-t}}{t}\,dt\,. \tag{18}$$

Note that when $\frac{r^2}{4\theta} \to \infty$, either when considering a large black hole ($r \to \infty$) or the commutative limit ($\theta \to 0$), we obtain the charged BTZ solution,

$$f^{\mathrm{BTZ}}\,(r) = -8M + \frac{r^2}{\ell^2} - \frac{Q^2}{8\pi^2}\ln|r|\,. \tag{19}$$

The line-element (14, 17) describes the geometry of a non-commutative black hole with the corresponding event horizons given by the following condition imposed on $f(r)$

$$f\,(r) = -8M\left(1 - e^{-r_\pm^2/4\theta}\right) + \frac{r_\pm^2}{\ell^2} - \\ - \frac{Q^2}{8\pi^2}\left[\,\ln|r_\pm| + \frac{1}{2}\,\mathrm{Ei}\Big(-\frac{r_\pm^2}{2\theta}\Big) - \mathrm{Ei}\Big(-\frac{r_\pm^2}{4\theta}\Big)\right]. \tag{20}$$

This equation cannot be solved in closed form. However, by plotting $f(r)$ one can see obvious intersections with the r-axis and determine numerically the existence of horizons and their radii. Fig. 1 shows that, instead of a single event horizon, there are different possibilities for this black hole:

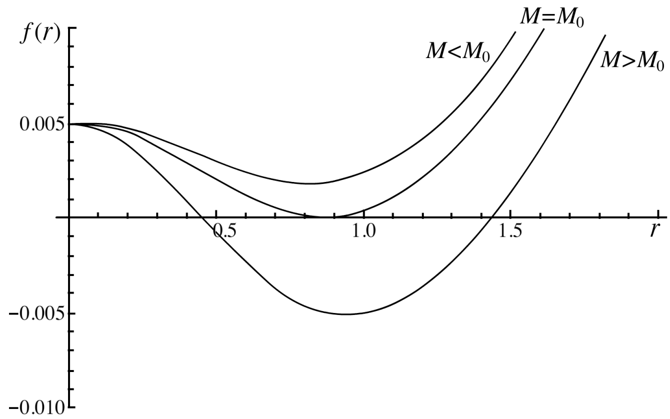

Fig. 1: Metric function f as a function of r. We have taken the values $\theta = 0.1$, $\ell = 10$ and $Q = 1$. The minimum mass is $M_0 \approx 0.00127$.

1) Two distinct horizons for $M > M_0$;
2) One degenerate horizon (extremal black hole) for $M = M_0$;
3) No horizon for $M < M_0$.

In view of this, there can be no black hole if the original mass is less than the lower-limit mass M_0. The horizon of the extremal black hole is determined by the conditions $f = 0$ and $\partial_r f = 0$, giving

$$\left[4\left(1 - e^{r_0^2/4\theta}\right)\theta + r^2\right]\left[\frac{1}{2}\,\mathrm{Ei}\left(-\frac{r_0^2}{2\theta}\right) - \mathrm{Ei}\left(-\frac{r_0^2}{4\theta}\right) + \ln|r_0| + \right.$$
$$\left. + \frac{2\theta}{r_0^2}\left(3 + e^{-r_0^2/2\theta} - 3e^{-r_0^2/4\theta} - e^{r_0^2/4\theta}\right)\right]^{-1} = \frac{Q^2\ell^2}{8\pi^2} \qquad (21)$$

and subsequently the mass of the extremal black hole can be written as

$$M_0 = \frac{\frac{2r_0^2}{\ell^2\theta} - \frac{Q^2}{8\pi^2\theta}\left(1 + e^{-r^2/2\theta} - 2e^{-r^2/4\theta}\right)}{4e^{-r_0^2/4\theta}}\,. \qquad (22)$$

§3. Thermodynamics. The Hawking temperature of the non-commutative black hole is

$$T_{\mathrm{H}} = \frac{1}{4\pi}\,\partial_r f|_{r_+} = \frac{r_+}{2\pi\ell^2}\left[1 - \frac{2M_{\mathrm{H}}\ell^2}{\theta}\,e^{-r_+^2/4\theta} - \right.$$
$$\left. - \frac{Q^2\ell^2}{16\pi^2 r_+^2}\left(1 + e^{-r_+^2/2\theta} - 2\,e^{-r_+^2/4\theta}\right)\right], \qquad (23)$$

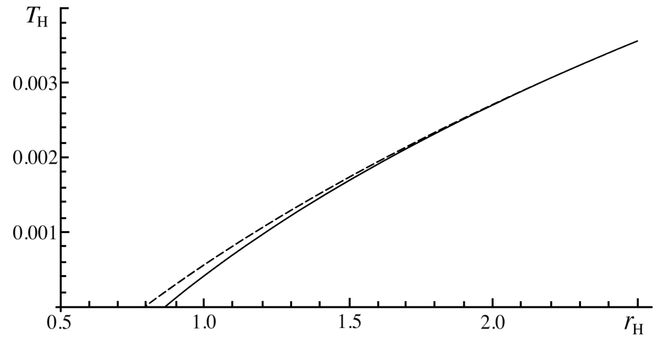

Fig. 2: The Hawking temperature versus r_H. The solid line represents the temperature for the non-commutative black hole with $\theta = 0.1$. There is no difference with respect to the charged BTZ solution (dashed line) for large r_H. In both cases, we use the values $\ell = 10$ and $Q = 1$.

where

$$M_H = \frac{r_+^2}{8\ell^2\left(1 - e^{-r_+^2/4\theta}\right)} - \frac{Q^2}{64\pi^2\left(1 - e^{-r_+^2/4\theta}\right)} \times \\ \times \left[\ln|r_+| + \frac{1}{2}\,\mathrm{Ei}\left(-\frac{r_+^2}{2\theta}\right) - \mathrm{Ei}\left(-\frac{r_+^2}{4\theta}\right)\right]. \quad (24)$$

For large black holes, i.e. $\frac{r_+^2}{4\theta} \gg 0$, one recovers the temperature of the rotating BTZ black hole,

$$T_H^{BTZ} = \frac{r_+}{2\pi\ell^2}\left(1 - \frac{Q^2\ell^2}{64\pi^2 r_+^2}\right). \quad (25)$$

As shown in Fig. 2, the Hawking temperature is a monotonically increasing function of the horizon radius for large black holes. For large black holes, there is indeed no difference with respect to the charged BTZ solution.

The first law of thermodynamics for a charged black hole reads

$$dM = T_H dS + \Phi\, dQ, \quad (26)$$

where the electrostatic potential is given by

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_{r_+} = -\frac{Q}{32\pi^2\left(1 - e^{-r_+^2/4\theta}\right)} \times \\ \times \left[\ln|r_+| + \frac{1}{2}\,\mathrm{Ei}\left(-\frac{r_+^2}{2\theta}\right) - \mathrm{Ei}\left(-\frac{r_+^2}{4\theta}\right)\right]. \quad (27)$$

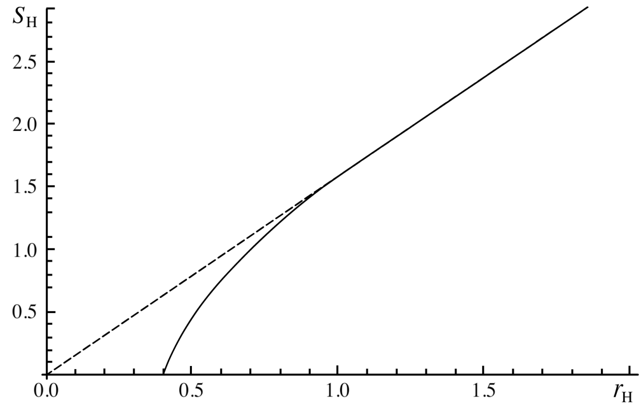

Fig. 3: Entropy versus $r_{\rm H}$. The solid line represents the entropy of the non-commutative black hole with $\theta = 0.1$. The dashed line represents the entropy of the charged BTZ black hole.

We calculate the entropy as

$$S = \int_{r_0}^{r_+} \frac{1}{T_{\rm H}}\, dM\,, \tag{28}$$

which gives

$$S = \frac{\pi}{2}\int_{r_0}^{r_+} \left(\frac{1}{1-e^{-\xi^2/4\theta}}\right) d\xi\,. \tag{29}$$

The entropy as a function of r_+ is depicted in Fig. 3. Note that, in the large black hole limit, the entropy function corresponds to the Bekenstein-Hawking entropy (area law), $S_{\rm BH} = \frac{\pi r_+}{2}$.

§4. Conclusion. We have constructed a non-commutative electrically charged black hole in AdS_3 space-time using an anisotropic perfect fluid inspired by the four-dimensional non-commutative black hole and a Gaussian distribution of electric charge. The result yields two horizons that degenerate into one in the extreme case. We have compared the thermodynamics of this black hole with that of a charged Banados-Teitelboim-Zanelli (BTZ) black hole. The Hawking temperature and entropy for a large non-commutative charged black hole approach those of the charged BTZ solution.

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