Charged Particle in the Extended Formulation of General Relativity

Patrick Marquet*

Abstract: In the recently presented extended formulation of General Relativity (the EGR theory), a "persistent field" expressed by a gravity-like energy-momentum tensor has been suggested. Due to the non-Riemannian curvature manifested by the theory, this field tensor is a true entity unlike Einstein's pseudo-tensor. Here this tensor is considered in the case of a charged particle in a gravitational field. In the "gravitational radiation damping", the usual relativistic treatment leads to a mass renormalization process. In the framework of the presented theory, this renormalization is not longer required.

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*Postal address: 7, rue du 11 nov, 94350 Villiers/Marne, Paris, France. E-mail: patrick.marquet6@wanadoo.fr. Tel: (33) 1-49-30-33-42.

Notations:

To completely appreciate this article, it is imperative to define some notations employed.

Indices. Throughout this paper, we adopt the Einstein summation convention whereby a repeated index implies summation over all values of this index:

4-tensor or 4-vector: small Latin indices $a, b, \ldots = 1, 2, 3, 4$;

3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3$;

4-volume element: d^4x;

3-volume element: d^3x.

Signature of space-time metric:

Hyperbolic $(+---)$ unless otherwise specified.

Operations:

Scalar function: $U(x^a)$;

Ordinary derivative: $\partial_a U$;

Covariant derivative in GR: ∇_a;

Covariant derivative in EGR: D_a or $'$, (alternatively).

Tensors:

Symmetrization: $A_{(ab)} = \frac{1}{2}!\,(A_{ab} + A_{ba})$;

Anti-symmetrization: $A_{[ab]} = \frac{1}{2}!\,(A_{ab} - A_{ba})$;

Kronecker symbol: $\delta_{ab} = (+1 \;\text{ if }\; a = b;\;\; 0 \;\text{ if }\; a \neq b)$;

Levi-Civita tensor: ϵ_{abcd} (where $\epsilon^{1234} = 0$).

Three-dimensional vectorial quantities:

$\boldsymbol{P} = P_\alpha$.

Introduction

As a follow up to my recent paper *The EGR Theory: An Extended Formulation of General Relativity* [1], we now turn to the consequences of this field contribution to an accelerated charged particle.

We recall the classical concept: In an electrostatic situation, the energy of a charged particle such as the electron, is $\frac{eV}{2}$, where V is the scalar potential of the field generated by the charge e. However, the Special Theory of Relativity tells us that any elementary particle is assumed to be a point mass or charge (non-elastic body), thus implying that at its "centre" $R=0$, where $V=\frac{e}{R}$ must become infinite. As a result, the proper energy (i.e. the proper mass of the electron) would also become infinite, which is physically irrelevant.

The usual way to overcome this difficulty leads to an implicit kind of external negative "mass" which compensates for the divergent one: this is accepted as the "renormalisation" process.

The free field predicted by the EGR theory is introduced in in the form of a "gravitational" energy-momentum tensor density $(\Im^{ab})_{\text{field}}$ next to the mass tensor density $(\Im^{ab})_{\text{mass}}$, which is the "continuation" of the classical energy-momentum pseudo-tensor so far associated with matter. In the framework of my understanding, this extra field, linked with the space-time segment curvature, naturally allows us to avoid the renormalisation requirement, providing the general electrodynamics with a clear and consistent explanation. The EGR Universe is entirely described by two curvatures. Accordingly, the present theory implicitly involves the EGR Ricci tensor R_{ab} rather than the Ricci tensor G_{ab}.

We begin this paper by recalling that, according to the Special Theory of Relativity, an accelerated electron will radiate and produce a reactive damping force in addition to the mechanical inertia force [2]. In the framework of the classical representation of the General Theory of Relativity (we will refer to it as GR), a charged particle does not suffer a reactive damping as long as its absolute acceleration is uniform. We may then expect that this particle actually radiates when deflected by a gravitational field i.e. when a kind of "Bremßtrahlung" effect takes a place; however, it has been shown that a more subtle phenomenon occurs. As has been pointed out by De Witt and Brehme [3], a plane or spherical sharp pulse of light when propagating in a curved 4-dimensional hyperbolic manifold, gradually develops a "tail" which is responsible for this electrogravitic "Bremßtrahlung". This "thinning out" of the elementary waves appears as an extra term in the relativistic equation of a moving charge [2].

Chapter 1. Relativistic Electrodynamics

§1.1. Electromagnetic radiation: variable fields

Variable potentials. Here, the charges are assumed to be located inside a volume element $d\vartheta$ where the variable charge density is $\mu(t)$.

Inside this volume, the scalar electrostatic potential V, which is derived from the electric field E, is

$$E = -\operatorname{grad} V. \tag{1.1}$$

Maxwell's second group of equations states that the variable field produced by arbitrary moving charges obeys the equation

$$\partial_b F^{ab} = -\frac{4\pi}{c}\, j^a, \tag{1.2}$$

$$j^a = \mu\,\frac{dx^a}{dt}\,, \tag{1.3}$$

where j^a is the four-vector density of charge μ. By setting the Lorentz gauge, $\partial_a A^a = 0$, we realise that

$$\frac{\partial^2 A^a}{\partial x^b\,\partial x_b} = \frac{4\pi}{c}\, j^a, \tag{1.4}$$

which can be decomposed into two equations

$$\Delta\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\,\boldsymbol{j} \tag{1.5}$$

and

$$\Delta V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 4\pi\mu\,. \tag{1.6}$$

If de is the variable charge in a given volume element $d\vartheta$, the charge density is

$$\mu = de\,(t)\,\delta R\,, \tag{1.7}$$

where δ is the Dirac function which will be analysed in the next section, while R is the distance from the origin of the coordinates, a unique point at which δR is not zero.

Retarded potentials. For an arbitrary charge distribution $\mu(x^a)$, we write

$$de = \mu\, d\vartheta\,.$$

For a volume ϑ we have

$$\mu = e\;\delta R\,.$$

In this case, equation (1.4) can be reduced to a plane wave equation whose solution is of the type

$$V = f\left(t - \frac{R}{c}\right).$$

This represents the progression of the potential V along R, however, with some retarded amplitude measured at the time t. This retarded amplitude results from the signal velocity limited by the light velocity c. Adding V_0 and A_0 to the solutions of equations (1.4) and (1.5), we have

$$\boldsymbol{A} = \frac{1}{c}\int \frac{\left(\boldsymbol{j}_{t-R/c}\right) d\vartheta}{R_a} + \boldsymbol{A}_0\,, \tag{1.8}$$

$$V = \int \frac{\left(\mu_{t-R/c}\right) d\vartheta}{R_a} + V_0\,. \tag{1.9}$$

If $R_a(t) = r_a - (r_a)_0$ is the distance to an electron e observed in $P(x_a)$ at t, the state of motion of the charge at an earlier time t' is determined by the equation

$$t' = t - \frac{R(t')}{c}\,. \tag{1.10}$$

In the resting frame at t', the field at $P(t)$ is simply given by the Coulomb potential

$$V = \frac{e}{c}\,(t - t') \quad \text{since} \quad \boldsymbol{A} = 0\,. \tag{1.11}$$

In a four-dimensional situation, in any arbitrary frame, we find the potential in the form

$$A^a = e\,\frac{u^a}{R_b}\,u^b, \tag{1.12}$$

which is the well-known expression of the Liénard-Wiechert potential, where

$$R_a = \left[\, c\,(t - t')\,,\; r_a - r'_a \,\right].$$

§1.2. Electromagnetic radiation: radiative damping

General coordinate system. On a general metric manifold, the dynamical equations for the electron of mass m_0 and charge e, in an electromagnetic field, are

$$m_0 \frac{\mathrm{D}u^a}{ds} = \frac{e}{c}\,F^{ab}u_b\,, \tag{1.13}$$

where $u^a = \frac{dx^a}{ds}$ is the four-velocity and the Maxwell tensor F_{ab} is

$$F_{ab} = \mathrm{D}_a A_b - \mathrm{D}_b A_a\,. \tag{1.14}$$

Here, the electromagnetic field's energy-momentum tensor is

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F^{c\cdot}_{\cdot a} + \frac{1}{4}\,g_{ab}\,F_{ck}F^{ck}\right). \tag{1.15}$$

In a general coordinate reference frame, we assume the following dynamical equations for a particle at z_a

$$\dot{z}^a = \frac{\mathrm{D}z^a}{d\tau} = u^b\,\mathrm{D}_b\,u^a, \tag{1.16}$$

$$\ddot{z}^a = \frac{\mathrm{D}\dot{z}^a}{d\tau} = \frac{d\dot{z}^a}{d\tau} + \Gamma^a_{bd}\,\dot{z}^b\,\dot{z}^d, \tag{1.17}$$

$$\dddot{z}^a = \frac{d\ddot{z}^a}{d\tau} + \Gamma^a_{bd}\,\ddot{z}^b\,\dot{z}^d, \tag{1.18}$$

where τ is the proper time of the particle*.

Three-dimensional radiative damping. An arbitrary distribution of charges with the velocities slow to c does not substantially vary during the time $\frac{R}{c}$. Therefore we expand $\mu_{t-R/c}$ and $\boldsymbol{j}_{t-R/c}$ into series of $\frac{R}{c}$.

Up to third order, we find for the scalar potential

$$V = -\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\mu\,d\vartheta\,. \tag{1.19}$$

Since the vector potential $\boldsymbol{A}$ already contains a term in $\frac{1}{c}$, we can restrict the expansion to second order. We take

$$\boldsymbol{A} = -\frac{1}{c^2}\,\partial_t f\int \boldsymbol{j}\,d\vartheta\,,$$

then follow with the transformations

$$\boldsymbol{A}' = \boldsymbol{A} + \operatorname{grad} f \quad \text{and} \quad V' = V - \frac{1}{c}\,\partial_t f\,.$$

Then we choose the function f so that the scalar potential V vanished, i.e.

$$f = -\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int R^2\mu\,d\vartheta\,.$$

Hence

$$\boldsymbol{A}' = -\frac{1}{c^2}\,\partial_t\int \boldsymbol{j}\,d\vartheta - \frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\,\operatorname{grad}\int R^2\mu\,d\vartheta\,.$$

*The connection coefficients (Christoffel symbols) are here assumed general for keeping the theory compatible with the EGR theory, we denote Γ^a_{bd} instead of the conventional Christoffel symbols $\left\{{a \atop bd}\right\}$ of General Relativity. See Page 178.

Thus, we arrive at the formula

$$\boldsymbol{A} = -\frac{1}{c^2}\,\partial_t\!\int \boldsymbol{j}\,d\vartheta - \frac{1}{3c^2}\frac{\partial^2}{\partial t^2}\int \boldsymbol{R}\mu d\vartheta\,. \tag{1.20}$$

The first-order terms of the field equation exhibit an additional force exerted on the charge. This force depends on the time derivative of the charge's acceleration. This force, resulting from a higher approximation, is called the *Lorentzian damping force*

$$F_\alpha = \frac{2}{3}\frac{e^2}{c^3}\,\dddot{z}_\alpha\,. \tag{1.21}$$

The equation of motion of the electron without external fields and solely subjected to (1.21), is due to the action of the charge itself

$$m_0\,\ddot{z}_\alpha = \frac{2}{3}\frac{e^2}{c^3}\,\dddot{z}_\alpha\,. \tag{1.22}$$

Ultrarelativistic case. In the Special Theory of Relativity, the equations of motion for the electron should be written

$$m_0\frac{du^a}{ds} = \frac{e}{c}\,F^{ab}u_b + f^a. \tag{1.23}$$

For the state of low velocity of the electron, the relation (1.23) should reduce to the expression (1.22). This condition is satisfied when

$$f^a = \frac{2}{3}\frac{e^2}{c}\left(\frac{d^2u^a}{ds^2} - u^a u^b\,\frac{d^2u_b}{ds^2}\right). \tag{1.24}$$

The second term in the brackets is chosen so as to satisfy the physical condition $f^a u_a = 0$, and so (1.24) can be written equally as

$$f^a = \frac{2}{3}\frac{e^2}{c^3}\left(\dddot{z}^a - \frac{1}{c^2}\,\dot{z}^a\,\ddot{z}^2\right). \tag{1.25}$$

Chapter 2. Trajectory of a Charged Particle in a Gravitational Field

§2.1. Brief reminder of the EGR theory

Free gravity field. In the EGR theory, the field equations

$$G_{da} = R_{da} - \frac{1}{2}\,g_{da}R$$

are generalized to

$$\mathsf{G}_{da} = \mathsf{R}_{da} - \frac{1}{2}\left(g_{da}\,\mathsf{R} - \frac{2}{3}\,J_{da}\right), \tag{2.1}$$

where $G_{da} = (G_{da})_{\mathrm{EGR}}$, The antisymmetric Ricci tensor $R_{da} = (R_{da})_{\mathrm{EGR}}$ is constructed with the general connection

$$\Gamma = \{\,\} + (\Gamma)_J \tag{2.2}$$

with the latter coefficients $(\Gamma)_J$, additional to the conventional Christoffel symbols $\{\,\}$, depending on the extra "segment curvature" through the 4-vector J according to

$$\mathrm{D}\,g_{ab} = \frac{1}{3}\left(g_{ac}\,J_b + g_{cb}\,J_a - g_{ab}\,J_c\right)dx^c.$$

The new generalized field equations are written down as

$$\mathcal{R}^{ab} + \mathcal{F}^{ab} = \varkappa\left[(\Im^{ab})_{\mathrm{mass}} + (\Im^{ab})_{\mathrm{field}}\right], \tag{2.3}$$

where the $(\Im^{ab})_{\mathrm{field}}$ represents the "energy-momentum" free field tensor density which is persistent even in the source-free EGR field equations.

Having defined the Lagrange density $\mathcal{H} = \mathfrak{R}^{ab} R_{ab}$ with

$$\mathfrak{R}^{ab} = \frac{\partial\mathcal{H}}{\partial R_{ab}}\,,$$

the free field density is inferred from the canonical equations

$$(\Im^a_b)_{\mathrm{field}} = \frac{1}{2\varkappa}\left[\mathcal{H}\,\delta^a_b - \partial_b\Gamma^e_{dk}\,\frac{\partial\mathcal{H}}{\partial\left(\partial_a\Gamma^e_{dk}\right)}\right] \tag{2.4}$$

(we have decomposed the curvature tensor density $\mathfrak{R}^{bc} = \sqrt{-g}\,R^{bc}$ into a symmetric part $\mathcal{G}^{bc}$ and an antisymmetric part $\mathcal{A}^{bc}$),

$$\mathfrak{R}^{bc} = \mathcal{G}^{bc} + \mathcal{A}^{bc} \quad \text{with} \quad \mathcal{G}^{bc} = \mathcal{R}^{bc} + \mathcal{E}^{bc}$$

so that

$$(\mathcal{E}^{bc})_{\prime,c} = 0 \quad \text{and} \quad \mathcal{J}^a = (\mathcal{A}^{ba})_{\prime,a} = \partial_a\mathcal{A}^{ba}$$

(due to the antisymmetry of $\mathcal{A}^{ba}$), we have a set of

$$\mathcal{J}^a = \sqrt{-g}\,J^a,$$

$$\mathcal{G}^{ab} = \sqrt{-g}\;g^{ab},$$

$$\mathcal{R}^{bc} = \sqrt{-g}\;R^{bc},$$

$$(\mathcal{G}^{bc})_{\prime,c} = -\frac{5}{3}\,\mathcal{J}^b,$$

$$(\mathcal{G}^{bc})_{\prime,a} = -\frac{1}{3}\,\delta^b_a\mathcal{J}^c + \delta^c_a\mathcal{J}^b,$$

where $'$, is the covariant derivative formed with Γ in (2.2). In fact, within the framework of my theory, the field equations (2.3) *always* have their second term, which corresponds to the free field tensor density

$$\mathcal{R}^{ab} + \mathcal{F}^{ab} = \varkappa\,(\Im^{ab})_{\text{field}}\,. \tag{2.5}$$

Thus, in the EGR theory, in the neighbourhood of matter, the mass density $(\Im^{ab})_{\text{mass}}$ increasingly dominates over the free field density $(\Im^{ab})_{\text{field}}$. This is the quasi-Riemannian regime of the classical theory.

Four-momentum vector of the free field. In tensor notation, we write the global four-energy momentum vector for the field and mass as

$$P^a = \frac{1}{c}\int \left[\,(T^{ab})_{\text{field}} + (T^{ab})_{\text{mass}}\right]\sqrt{-g}\;dS_b$$

across any hypersurface. Inspection shows that the pseudo-tensor $\Im^{ab}$ is a true tensor quantity lending support to the theory of a free field (which is merely the natural extension of the Riemannian gravitational field), for which the quantity is classically attributed to the mass. When integration is performed on the volume ϑ containing this mass, the tensor field $(T_{ab})_{\text{field}}$ vanishes inside the matter, thus only the time component of the four-momentum vector remains (i.e. we are in the Riemannian regime)

$$P^4 = m_0 c = \frac{1}{c}\int \left[-(T^a_a)_{\text{mass}}\right]\sqrt{-g}\;d\vartheta$$

or

$$m_0 c^2 = \int \left[(T^1_1)_{\text{mass}} + (T^2_2)_{\text{mass}} + (T^3_3)_{\text{mass}} - (T^4_4)_{\text{mass}}\right]\sqrt{-g}\;d\vartheta$$

that is the total mass of the given corpuscle (particle). If distantly located from the source, $(\Im^{ab})_{\text{mass}} \to 0$ and

$$P^a \approx \frac{1}{c}\int (\Im^{ab})_{\text{field}}\,dS_b\,. \tag{2.6}$$

§2.2. Gravitational influence

§2.2.1. Dirac bi-tensors

Dirac's distribution function. We now consider the *delta function* introduced by P. A. M. Dirac [4]

$$\delta(x' - x)\,, \tag{2.7}$$

which is known as the Dirac distribution function

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0\,,\ \delta(0) = \infty\,, \tag{2.8}$$

hence

$$\int_{-\infty}^{+\infty} dx = 1\,. \tag{2.9}$$

If $f(x)$ is a continuous function at the point $x = 0$, we have

$$\int_{-\infty}^{+\infty} \delta(x)\, f(x)\, dx = f(0) \tag{2.10}$$

under a more general form

$$\int \delta(x-a)\, f(x)\, dx = f(a)\,, \tag{2.11}$$

where the integration domain contains the point $x=a$, and $f(x)$ is continuous at the point $x = a$.

We write (2.11) as

$$\langle \delta(x,x'),\ f(x') \rangle = f(x)\,. \tag{2.12}$$

The notation

$$\delta(x,x') \tag{2.13}$$

is called the *Dirac bi-scalar*. It will be generalized in the next section.

Displacement bi-tensors. On a differential manifold V_n, we are going to consider a point x' located in the neighbourhood of another point x. Along the geodesic connecting x' to x, we define a "displacement" which represents a "canonical isomorphism" (basis-independent) of the tangent space T_x at x on the manifold, into the tangent space $\mathrm{T}_{x'}$ at x'. The free bases $e_a(x)$ and $e_c(x')$ are attributed to the neighbourhoods.

The relevant isomorphism therefore defines a "bi-tensor", which we call a *displacement tensor*, and denote as

$$\mathfrak{t}_a^{c'}\,, \tag{2.14}$$

hence

$$g_{ab}\, \mathfrak{t}^a_{c'} \mathfrak{t}^b_{d'} = g_{c'd'}\,. \tag{2.15}$$

Here we have

$$\mathfrak{t}_{ac'} = g_{c'd'}\, \mathfrak{t}_a^{d'} = g_{ab}\, \mathfrak{t}^b_{c'}\,, \quad (\mathfrak{t}_a^{d'} \mathfrak{t}^b_{d'} = \mathfrak{t}^d_{a'} \mathfrak{t}_d^{b'} = \delta^b_a)\,, \tag{2.16}$$

$$\mathfrak{t} = \det \|\, \mathfrak{t}_{ac'}(x) \| = \sqrt{\det \|\, g_{ab}(x) \|}\, \sqrt{\det \|\, g_{c'd'}(x) \|}\ , \tag{2.17}$$

the particular case $x = x'$ implies

$$\mathfrak{t}_a^{c'}(x, x'=x) = \delta_a^{c'} \quad \text{or} \quad \mathfrak{t}_{ac'}(x,x'=x) = g_{ac'}\,, \tag{2.18}$$

$$\nabla_a\, \mathfrak{t}_c^{d'}(x,x'=x) = \nabla_{c'}\, \mathfrak{t}_a^{d'}(x,x'=x) = 0\,. \tag{2.19}$$

If V_n is an Euclidean space of the given signature (e.g. Minkowskian), we simply have

$$\mathfrak{t}_{ac'} = e_a \, e_{c'} \,. \tag{2.20}$$

We choose the space-time signature, as earlier, to be

$$g_{ab} = \mathrm{diag}\,(+\,-\,-\,-)\,, \tag{2.21}$$

so the determinant is $g = \det \| g_{ab} \| < 0$, while $\sqrt{-g} > 0$.

§2.2.2. The Feynman propagator (reminder)

The Pauli-Jordan propagator (reminder). In the quantized field technique, the commutation function of the scalar field is introduced. It satisfies

$$\mathcal{D}(x) = \mathcal{D}^{+}(x) + \mathcal{D}^{-}(x) \tag{2.22}$$

with

$$\begin{aligned} \mathcal{D}^{+}(x) &= -\mathcal{D}^{-}(-x) = \\ &= \frac{1}{(2\pi)^3}\, i \int \big[\exp\,(i\,Px)\big]\, \delta(P^2 - m_0^2)\, \theta(P^4)\, d^3P \,. \end{aligned} \tag{2.23}$$

This commutation function or the *Pauli-Jordan propagator* is explicitly written

$$\mathcal{D}(x, x') = \frac{1}{(2\pi)^3}\, i \int \big[\exp\,(i\,Px)\big]\, \epsilon(P^4)\, \delta(P^2 - m_0^2)\, d^3P \,, \tag{2.24}$$

where

$$\epsilon(P^4) = \theta(P^4) - \theta(-P^4)$$

is the "sign function"

$$\epsilon = +1 \quad \text{for} \quad P^4 > 0\,,$$

$$\epsilon = -1 \quad \text{for} \quad P^4 < 0\,.$$

The upper indices + and − indicate, respectively, the positive or negative energy parts contributed into the complete commutator $\mathcal{D}$, which corresponds to the future and the past, and whose boundaries are the characteristic hyperboloids in the Minkowski representation.

The Green function. The Jordan-Pauli commutation relation is an odd function

$$\mathcal{D}(x, x') = -\,\mathcal{D}(x', x)\,. \tag{2.25}$$

This (scalar) propagator is Lorentz invariant. It satisfies the homogeneous Klein-Gordon equation

$$\big(-\,\partial^a \partial_a - m_0^2\big)\, \mathcal{D}(x) = 0\,. \tag{2.26}$$

We then define the *Green function* of the scalar field by the equation

$$(-\,\partial^a\partial_a - m_0^2)\,\mathbb{G}(x,x') = -\,\delta^{(4)}(x,x')\,, \tag{2.27}$$

and then, passing to the momentum representation

$$\mathbb{G}(x,x') = \frac{1}{(2\pi)^4}\int \big[\exp\big(i\,P(x,x')\big)\big]\,\mathbb{G}(P)\,d^4P\,, \tag{2.28}$$

we obtain, for $\mathbb{G}(P)$, the expression

$$\mathbb{G}(P) = \frac{1}{m_0^2 - P^2}\,. \tag{2.29}$$

Writting the denominator as

$$P^2 = (P^4)^2 - (\mathcal{P}^2 + m_0^2)\,, \tag{2.30}$$

we see that for a given $\mathcal{P}^2$, the time component P_4 has two poles

$$P_4 = \pm E\,, \tag{2.31}$$

where the total energy of the particle is

$$E = \sqrt{P^2 + m_0^2}\,. \tag{2.32}$$

In order to remove this ambiguity when integrating (2.29) over d^4P, the Feynman contour rules should be used to circumvent the poles. First we consider the "retarded" Green function defined by the condition

$$\mathbb{G}^-(x,x') = 0 \quad \text{for} \quad x^4 - x'^4 < 0\,. \tag{2.33}$$

We then remark that the function (2.29) is not substantially modified, if multiplied by $\exp\big[-\,\epsilon(x^4 - x'^4)\big]$, where $\epsilon > 0$,

$$\mathbb{G}^-\exp\big[-\,\epsilon(x^4 - x'^4)\big] = \mathbb{G}_\epsilon\,, \tag{2.34}$$

and it can thus be represented by

$$\mathbb{G}^-(x,x') = \lim_{\epsilon\to 0}\mathbb{G}_\epsilon\,, \tag{2.35}$$

and $\mathbb{G}_\epsilon$ is defined by (2.31), hence satisfies

$$\big[\Delta - (\partial_t + \epsilon)^2 - m_0^2\big]\,\mathbb{G}_\epsilon = -\,\delta(x,x')\,. \tag{2.36}$$

In the momentum representation, when $\epsilon \to 0$, we have

$$\mathbb{G}_\epsilon = \frac{1}{m_0^2 - (P^4 - i\epsilon)^2 + \mathcal{P}^2} \longrightarrow \frac{1}{m_0^2 - P^2 - 2i\epsilon\mathcal{P}^4}\,, \tag{2.37}$$

and (2.31) takes the form

$$\mathbb{G}^{-}(x, x') = \frac{1}{(2\pi)^4} \int \frac{\exp\left[iP(x, x')\right]}{m_0^2 - P^2 + 2i\epsilon P^4}\, d^4P\,. \tag{2.38}$$

The same effect can be achieved if one integrates along the real axis by shifting the poles by an infinitesimal mass of the particle in the complex plane.

In the same way, the advanced Green function defined by

$$\mathbb{G}^{+}(x, x') = 0\,, \quad \text{for} \quad x^4 - x'^4 > 0\,, \tag{2.39}$$

which satisfies (3.36), is of the form

$$\mathbb{G}^{+}(x, x') = \frac{1}{(2\pi)^4} \int \frac{\exp(iPx)}{m_0^2 - P^2 + 2i\epsilon P^4}\, d^4P\,. \tag{2.40}$$

The integral

$$\mathbb{G}(x) = \frac{1}{(2\pi)^4} \int \frac{\exp(iPx)}{m_0^2 - P^2}\, d^4P \tag{2.41}$$

can be taken over the principal value, upon being separated into real and imaginary parts

$$\frac{1}{x + i\epsilon P^4} = \frac{P}{x} - i\pi\,\delta(x)\,\epsilon(P^4)\,, \tag{2.42}$$

and we obtain

$$\frac{P}{(2\pi)^4} \int \frac{\exp(iPx)}{m_0^2 - P^2}\, d^4P = \frac{1}{2}\,\epsilon(x^4)\,\mathbb{G}(x)\,. \tag{2.43}$$

Local bi-tensors on a four-dimensional manifold. Let us recall that the simplest example of a bi-tensor is the product of two local vectors taken at the different space-time points x and x'

$$A^k(x) \quad \text{and} \quad B_a(x')\,, \tag{2.44}$$

$$C^k_a(x, x') = A^k(x)\, B_a(x')\,. \tag{2.45}$$

We shall here adopt De Witt's convention that indices taken from the Latin characters $a \ldots k$ are always to be associated with the point x' (denoted, from now, by z), while indices taken from k to y, are always associated with the point x.

The transformation law for the bi-tensor (2.45) is given by

$$C'^k_a = \left(\frac{\partial x'^k}{\partial x_m}\right)\left(\frac{\partial z^b}{\partial z'^a}\right) C^m_b\,. \tag{2.46}$$

The Dirac bi-scalar (2.13) extended to the Minkowski space is

$$\delta^{(4)}(x,z) = \delta(x^4-z^4)\,\delta(x^1-z^1)\,\delta(x^2-z^2)\,\delta(x^3-z^3) = \\ = \delta^{(4)}(z,x)\,, \qquad (2.47)$$

and is also called the *bi-density.*

We also define the geodesic interval bi-scalar $s\,(z,x)$ by the invariant

$$g^{km}\,\delta_k s\,\delta_m s = g^{ab}\,\partial_a s\,\partial_b s = \pm 1 \qquad (2.48)$$

with

$$\lim_{x\to z} s = 0\,.$$

§2.2.3. Trajectory of a charged particle

World tube. Let us consider a particle describing a world line whose point coordinate will always be denoted by z. We construct a small sphere surrounding the particle. The energy-momentum flow will be determined across the surface. In the course of time, such a sphere generates a hypersurface called a world tube.

We begin by introducing, at the point z on the world line of the particle, three unit vectors orthogonal to each other and to the world line itself

$$n^a_\alpha\, n_{\beta a} = \delta_{\beta\alpha}\,, \qquad n_{\alpha a}\,\dot{z}^a = 0\,. \qquad (2.49)$$

We next introduce a set of direction cosines ς satisfying

$$\varsigma_\alpha\,\varsigma_\alpha = 1 \qquad (2.50)$$

in terms of which we can specify the direction, relative to n^a_α, of an arbitrary unit vector perpendicular to the world line at z.

Then, in the direction of this arbitrary vector, we construct a geodesic from z extending throughout a fixed distance ξ to a point x of the "tube wall". The coordinates at the point z depend on the direction cosines ς_α and on the proper time τ at this point, which is explicitly expressed at the tube wall by the function $x^k(\varsigma,\tau)$.

Let us set up a bi-scalar σ related to the distance ξ as

$$\sigma = \frac{1}{2}\,\xi^2,$$

whence

$$\partial_a\sigma = -\,\xi\, n_{\alpha a}\,\varsigma^\alpha\,, \qquad (\partial_a\sigma)\,\dot{z}^a = 0\,. \qquad (2.51)$$

A pair of independent variations $\delta_1\varsigma_\alpha$, $\delta_2\varsigma_\alpha$ in the direction cosines defines an element $d\Omega$ of solid angle by the relation

$$\varsigma_\alpha d\Omega = \epsilon_{\alpha\beta\gamma}\,\delta_1\varsigma^\beta\,\delta_2\varsigma^\gamma\,,$$

or in virtue of (2.51), we have

$$\partial_q \partial_a \sigma \, \delta_3 \, x^q = \varsigma_a \, d\xi \,,$$

or

$$\delta_3 \, x^q = -(D^{qa})^{-1} \varsigma_a \, d\xi \,, \qquad D_{qa} = -\, \partial_q \partial_a \, \sigma \,.$$

We define a "tube section" as

$$dS_q = \epsilon_{qruw} \, \delta_1 x^r \, \delta_2 \, x^u \, \delta_3 \, x^w$$

and with $\Delta = -\mathfrak{t}^{-1} \det \| - D_{qa} \|$, where $\mathfrak{t}$ is the determinant (2.17), we obtain

$$dS_q = -\frac{1}{c\sqrt{-g}} \, \Delta^{-1} \, D_{qa} \, \dot{z}^a \, \xi^2 \, d\xi \, d\Omega \,. \tag{2.52}$$

§2.2.4. Dynamical equations for a particle

The conserved energy-momentum tensor. Let L denotes the surface of the world tube limited by two sections of hypersurfaces S_1 and S_2 corresponding to two proper times τ_1 and τ_2 (with $\tau_1 < \tau_2$).

We choose the integration volume d^4x as a portion of the tube, in order to express an integral conservation condition for the energy-momentum bi-tensor density $\Im^{qr}$.

However, one cannot integrate the divergence of $\Im^{qr}$ over the four-volume (at x) d^4x, to replace the volume integral by an integral over the hypersurface S_r containing z, since Gauss' theorem is not longer applicable for a bi-tensor.

There is nevertheless a natural procedure to overcome this difficulty by introducing the displacement bi-tensor $\mathfrak{t}^a_q$ in order to refer to the contributions into the integral

$$I^a = \int (\mathfrak{t}^a_q \, \partial_r \Im^{qr}) \, d^4 r \tag{2.53}$$

at the point x back to some fixed point z.

The latter integral becomes a local four-vector at z where

- x_a corresponds to x_q, and
- x'_a corresponds to z_a.

Let us then consider the integral over S_1, S_2 and the volume ϑ, the conservation condition for $\Im_{ab}$ is then written down as

$$\frac{1}{c} \int (\mathfrak{t}^a_q \, \partial_r \Im^{qr}) \, d^4 x = 0 \,, \tag{2.54}$$

Integrating by parts

$$\frac{1}{c}\left(\int_L + \int_{S_1} + \int_{S_2}\right) \mathfrak{t}^a_q \, \Im^{qr} \, dS_r - \frac{1}{c}\int_\vartheta (\partial_r \mathfrak{t}^a_q) \, \Im^{qr} \, d^4x = 0 \tag{2.55}$$

with zero contribution of the last integral, and considering the replacement

$$\int_L \longrightarrow \int_{\tau_1}^{\tau_2} \int_{4\pi} \,, \tag{2.56}$$

we can write (2.55) in the limit $\xi \to 0$, while taking (2.52) into account,

$$\lim_{\xi\to 0} \, \frac{1}{c} \int_{\tau_1}^{\tau_2}\!\!\int_{4\pi} \mathfrak{t}^a_q \, \Im^{qr} dS_r + m_0 \left[\, \mathfrak{t}^a_{b'} \big(z(\tau'), z(\tau)\big) \, \dot{z}^{b'}(\tau')\right]_{\tau=\tau_1}^{\tau'=\tau_2} -$$

$$- \, m_0 \int_{\tau_1}^{\tau_2} \partial_{r'} \mathfrak{t}^a_{b'} \big(z(\tau'), z(\tau)\big) \, \dot{z}^{b'} \dot{z}^{r'}(\tau') \, d\tau' = 0 \,. \tag{2.57}$$

The next step is to let τ_1 and τ_2 both approach τ, and denoting their infinitesimal separation in the limit by $d\tau$, we express the relation (2.57) as follows

$$m_0 \, \ddot{z}_a \, d\tau = - \lim_{\xi\to 0} \, \frac{1}{c} \int \mathfrak{t}^a_q \, \Im^{qr} dS_r \,. \tag{2.58}$$

The geodesic principle is obviously given by

$$m_0 \, \ddot{z}_a = 0 \,. \tag{2.59}$$

In the framework of the Euclidean approximation, when the particle's trajectory is taken along x, the latter equation reduces to

$$m_0 \, \frac{d^2x}{d\tau^2} = 0 \,. \tag{2.60}$$

Chapter 3. Gravitational Damping

§3.1. Green functions on a curved manifold

§3.1.1. Scalar Green functions

Elementary solutions of J. Hadamard. In a non-Euclidean space, the second derivatives of any vector or tensor are not equivalent

$$(\mathrm{D}_e \mathrm{D}_k - \mathrm{D}_k \mathrm{D}_e) \, A^{h\cdots}_{d\cdots} = - \, R^{h\cdots}_{\cdot iek} \, A^{i\ldots}_{d\ldots} - 2\Gamma^i_{ke} \mathrm{D}_i A^{h\cdots}_{d\cdots} + R^{\,i\,\cdots}_{\cdot\, dke} \, A^{h\ldots}_{i\cdots} \,. \tag{3.1}$$

From the identities, the equations

$$- \, \mathrm{D}_e F^{de} = j^d \tag{3.2}$$

read

$$-\sqrt{-g}\; g^{dh} g^{ek} \mathrm{D}_e \left(\mathrm{D}_h A_k - \mathrm{D}_k A_h\right) =$$
$$= \sqrt{-g}\; g^{ek} \mathrm{D}_e \mathrm{D}_k A^d - \sqrt{-g}\; \mathrm{D}^d \mathrm{D}_k A^k - \sqrt{-g}\; R^{cd} A_c = -\, j^d, \quad (3.3)$$

and by fixing a gauge

$$\mathrm{D}_k A^k = 0\,, \quad (3.4)$$

we have

$$\sqrt{-g}\; \left(g^{ek} \mathrm{D}_e \mathrm{D}_k A^d - R^{dh} A_h\right) = -\, j^d. \quad (3.5)$$

Consider then the vector wave equation

$$g^{ek} \mathrm{D}_e \mathrm{D}_k A^d - R^{dh} A_h = 0\,. \quad (3.6)$$

Following Hadamard, we shall try to find so called "elementary solutions" corresponding to Green functions. We can then infer the particular solutions of (3.5).

The Feynman propagator. We first consider here the scalar wave equation on a four-dimensional manifold

$$g^{dh} \partial_d \partial_h A = 0\,. \quad (3.7)$$

Here, we find the elementary solution which is a bi-scalar having the form

$$\mathbb{G}^{(1)} = \frac{1}{(2\pi)^2} \left(\frac{\mathfrak{u}}{\xi} + \mathfrak{b} \ln|\xi| + \mathfrak{w} \right), \quad (3.8)$$

where $\mathfrak{u}$, $\mathfrak{b}$, $\mathfrak{w}$ are bi-scalars satisfying the normalization condition

$$\lim_{x \to z} \mathfrak{u} = 1\,. \quad (3.9)$$

After some algebra, we show the validity of the equation

$$\mathfrak{u}^{-1} \partial_d \mathfrak{u} = \frac{1}{2}\, \Delta^{-1} \partial_d \Delta\,, \quad (3.10)$$

which, with the boundary condition (3.9), has the unique solution

$$\mathfrak{u} = \sqrt{\Delta}\;. \quad (3.11)$$

Eventually we arrive at

$$\lim_{x \to z} \mathfrak{b} = \frac{1}{12}\, \mathbb{G}\,. \quad (3.12)$$

Separating the full Green function $\mathbb{G}^{\mathrm{F}}$ into real and imaginary parts

$$\mathbb{G}^{\mathrm{F}} = \mathbb{G}^{(1)} - 2i\, \mathbb{G}\,, \quad (3.13)$$

where

$$\mathbb{G}^{\mathrm{F}} = \frac{1}{(2\pi)^2}\left[\frac{\sqrt{\Delta}}{(\xi + i0)} + \mathfrak{b}\ln(\xi + i0) + \mathfrak{w}\right] \tag{3.14}$$

is identified with the Feynman propagator.

The formula (2.42) becomes

$$\frac{1}{\xi + i0} = \frac{\mathfrak{p}}{\xi - \pi i \delta(\xi)} \tag{3.15}$$

and

$$\ln\left(\xi + i0\right) = \ln|\xi| + \pi i \epsilon(-\xi) \tag{3.16}$$

with the sign function such that

$$\epsilon(\xi) = 0 \quad \text{for} \quad \xi < 0$$

and

$$\epsilon(\xi) = 1 \quad \text{for} \quad \xi > 0\,.$$

The scalar Green function corresponding to the bi-scalar $\mathfrak{b}$ can be computed as

$$\mathbb{G} = \frac{1}{8\pi}\left[\sqrt{\Delta}\;\delta(\xi) - \mathfrak{b}\,\epsilon(-\xi)\right]. \tag{3.17}$$

§3.1.2. Vector Green functions

Hadamard solutions. Consider now the wave equation

$$g^{hk}\,\mathrm{D}_h\mathrm{D}_k A_d + R_{d\cdot}^{\cdot h} A_h = 0\,. \tag{3.18}$$

The procedure is entirely analogous to the above, thus we introduce the elementary solution of the form

$$\mathbb{G}^{(1)}_{qa} = \frac{1}{(2\pi)^2}\left[\frac{\mathfrak{u}_{qa}}{\xi + \mathfrak{b}_{qa}}\ln|\xi| + \mathfrak{w}_{qa}\right], \tag{3.19}$$

where the functions $\mathfrak{u}_{qa}$, $\mathfrak{b}_{qa}$ and $\mathfrak{w}_{qa}$ are now bi-vectors. Normalization for $\mathfrak{u}_{qa}$ leads to

$$\lim_{x\to z}\mathfrak{u}_{qa}(x,z) = g_{qa}(z)$$

and, after some algebra, we find

$$\mathfrak{u}_{qa} = \sqrt{\Delta}\;\mathfrak{t}_{qa}\,. \tag{3.20}$$

Making use of the extension, for the bi-vector $\mathfrak{b}_{qa}$,

$$\mathfrak{b}_{qa} = \mathfrak{t}_{qa}\left(1 - \frac{1}{12}\,R^{be}\,\partial_b\sigma\,\partial_e\sigma + O(s^2)\right) \tag{3.21}$$

at the limit

$$\lim_{x\to z} \mathfrak{b}_{qa} = -\frac{1}{2}\,\mathfrak{t}^{b}_{q}\left(R_{ab} - \frac{1}{6}\,g_{ab}R\right). \tag{3.22}$$

The presence of the determinant (Δ symbol) in (3.20) reveals the singular behaviour of the elementary waves originating from the point z: this represents actually the so-called "thinning out" of these waves due to the induced curvature.

The Feynman propagator. The full propagator is of the form

$$\mathbb{G}^{\mathrm{F}}_{qa} = \mathbb{G}^{(1)}_{qa} - 2i\,\mathbb{G} \tag{3.23}$$

that is

$$\mathbb{G}^{\mathrm{F}}_{qa} = \frac{1}{(2\pi)^2}\left[\sqrt{\Delta}\,\frac{\mathfrak{t}_{qa}}{(\xi + i0) + \mathfrak{b}_{qa}}\ln(\xi + i0) + \mathfrak{w}_{qa}\right]. \tag{3.24}$$

Advanced or retarded Green functions. We set

$$\mathbb{G}^{\pm}_{qa} = \int \sqrt{\mathfrak{t}}\;\mathbb{G}^{\pm}_{qr'}\,\mathfrak{t}^{r'}_{a}\,\delta^{(4)}d^4x'. \tag{3.25}$$

The quantities $\mathbb{G}^{\pm}_{qa}$ correspond to advanced and retarded portions of the Green functions $\mathbb{G}_{qa}$, whose components depend on two distinct points x and z: they define a bi-vector.

If we consider an arbitrary space-like hypersurface $S(x)$ containing x, we regard "actions" as retarded when the source z^a lies to the past of S, and advanced when the source z^a lies to the future of S. The "symmetric" Green function is then

$$\mathbb{G}_{qa} = \frac{1}{8\pi}\left[\sqrt{\Delta}\;\mathfrak{t}_{qa}\,\delta\xi - \mathfrak{b}_{qa}\,\epsilon(-\xi)\right], \tag{3.26}$$

where the functions $\mathbb{G}$ can, just as in the flat-space case, be separated into advanced and retarded parts

$$\mathbb{G}_{qa} = \frac{1}{2}\left(\mathbb{G}^{-}_{qa} + \mathbb{G}^{+}_{qa}\right) \tag{3.27}$$

with

$$\mathbb{G}^{-}_{qa} = 2\,\epsilon(S,\,z)\,\mathbb{G}_{qa}(x,z)\,, \tag{3.28}$$

$$\mathbb{G}^{+}_{qa} = 2\,\epsilon(z,\,S)\,\mathbb{G}_{qa}(x,z)\,, \tag{3.29}$$

$$\epsilon\left[S(x),\,z\right] = 1 - \epsilon\left[z,\,S(x)\right] = 1\,,$$

when z lies to the past of S, and vanishes when it lies to the future.

§3.2. Dynamical equations for the electron

§3.2.1. Tensor density of the electromagnetic field

Energy-momentum field global tensor density. On an arbitrary manifold, the approximated Lagrangian for a particle of mass m_0 is

$$L_m = -\, m_0 c^2 \int \sqrt{-\, g_{ab}\, \dot{z}^a\, \dot{z}^b}\ \delta^{(4)}\, d\tau\,. \tag{3.30}$$

The inferred massive tensor density of this particle with respect to the proper time τ following the geodesic $z(\tau)$ is

$$M^{qr} = m_0 c \int \sqrt{\mathfrak{t}}\ \mathfrak{t}^q_a\, \mathfrak{t}^r_b\, \dot{z}^a\, \dot{z}^b\, \delta^{(4)}\, d\tau\,. \tag{3.31}$$

For an electron interacting with an electromagnetic field, the Lagrangian density becomes

$$L = -\, m_0 c^2 \int \sqrt{-\, g_{ab}\, \dot{z}^a\, \dot{z}^b}\ \delta^{(4)}\, d\tau + \\ + e \int A_a\, \dot{z}^a\, \delta^{(4)}\, d\tau - \frac{1}{16\pi} \sqrt{-g}\ F_{qr} F^{qr}. \tag{3.32}$$

The current vector density expressed with the charge density (1.7) can be determined from the four-velocity $\dot{z}^a$ at the point z, by parallel displacement along the geodesic extending z to x

$$j^q = e \int \sqrt{\mathfrak{t}}\ \mathfrak{t}^q_a\, \dot{z}^a\, \delta^{(4)}\, d\tau\,. \tag{3.33}$$

The form of this density justifies the form of the second term in (3.32), which corresponds to the classical electron-field interaction, $eA_q j^q$. Application of the least action principle to

$$S = \frac{1}{c} \int L\, d^4x \tag{3.34}$$

yields the dynamical equations

$$m_0\, \ddot{z}^a = \frac{e}{c}\, F^{a\cdot}_{\cdot b}\, \dot{z}^b, \tag{3.35}$$

$$\sqrt{-g}\ \partial_r F^{qr} = \frac{4\pi}{c}\, j^q = \partial_r F^{qr}. \tag{3.36}$$

Given the current density j^q, the tensor $F^{a\cdot}_{\cdot b}$ appearing on the right hand side of (3.32) is divergent, and this leads to the well-known difficulty that the electron's proper mass m_0 is infinite, which must thereby be renormalized.

For reasons which will become clear later, we hereby proceed to consider the tensor density of the whole system as

$$\Im^{ab} = (\Im^{ab})_{\text{mass}} + (\Im^{ab})_{\text{field}} + (\Im^{ab})_{\text{elec}}\,. \tag{3.37}$$

Advanced or retarded potentials. According to Quantum Electrodynamics, the particular solutions of equation (3.5) are the retarded and/or advanced potentials

$$A^{-}_{q}(x) = \frac{4\pi}{c}\int \mathbb{G}^{-}_{qr}(x,x')\, j^{r'}(x')\, d^4x', \tag{3.38}$$

$$A^{+}_{q}(x) = \frac{4\pi}{c}\int \mathbb{G}^{+}_{qr}(x,x')\, j^{r'}(x')\, d^4x'. \tag{3.39}$$

Substituting the expressions of j^q in the previous equations, we obtain

$$A^{\pm}_{q} = 4\pi e\int_{-\infty}^{+\infty} \mathbb{G}^{\pm}_{qa}\, \dot{z}^a\, d\tau =$$

$$= \pm\, e\int_{\tau_s}^{\pm\infty} \left[\mathfrak{u}_{qa}\,\delta\xi - \mathfrak{b}_{qa}\,\epsilon(-\xi)\right] \dot{z}^a\, d\tau\,, \tag{3.40}$$

where τ_{S} is the value of the proper time at the point of intersection of the world line of the particle with an arbitrary hypersurface $S(x)$ containing x.

Defining the advanced and retarded proper time of the particle relative to the point x, $\tau_{\pm}$, we obtain the advanced and retarded potentials as

$$A^{\pm}_{q} = \mp\, e\left[\sqrt{\Delta}\;\mathfrak{t}_{qa}\,\dot{z}^a\left(\dot{z}^b\,\partial_b\,\xi\right)^{-1}\right]_{\tau=\tau^{\pm}} \mp\, e\int_{\tau^{\pm}}^{\infty} \mathfrak{b}_{qa}\,\dot{z}^a\, d\tau\,. \tag{3.41}$$

These are the covariant Liénard-Wiechert potentials. For flat spacetime these potentials obviously reduce to the form (1.12).

Retarded and advanced fields. From the potentials defined above, we define the corresponding proper fields

$$F^{\pm}_{qr} = \partial_q\, A^{\pm}_{r} - \partial_r\, A^{\pm}_{q}. \tag{3.42}$$

The total field can be expressed in the alternative forms

$$F_{qr} = (F_{qr})^{\text{in}} + F^{-}_{qr} = (F_{qr})^{\text{out}} + F^{+}_{qr}\,, \tag{3.43}$$

where *in* and *out* mean the incoming field and the outgoing field, respectively.

Defining the average field

$$\langle F_{qr}\rangle = \frac{1}{2}\big(F^{-}_{qr} + F^{+}_{qr}\big)\,, \tag{3.44}$$

we write the total field in terms of the average free non-radiative field

$$F_{qr} = \langle (F_{qr})^{\text{free}}\rangle + \langle F_{qr}\rangle \tag{3.45}$$

with

$$\langle (F_{qr})^{\text{free}}\rangle = \frac{1}{2}\left[(F_{qr})^{\text{in}} + (F_{qr})^{\text{out}}\right]. \tag{3.46}$$

The field strengths can be explicitly written as

$$F^{\pm}_{qr} = \mp\, e\,\Big\{(\mathfrak{u}_{ra}\partial_q\xi - \mathfrak{u}_{qa}\partial_r\xi)\,\dot{z}^a\,(\dot{z}^b\dot{z}^e\partial_b\partial_e\xi + \ddot{z}^b\partial_b\xi)(\dot{z}^d\partial_d\xi)^{-3} - \\ - \left[\partial_b(\mathfrak{u}_{ra}\partial_q\xi - \mathfrak{u}_{qa}\partial_r\xi)\,\dot{z}^a\dot{z}^b + (\mathfrak{u}_{ra}\partial_q\xi - \mathfrak{u}_{qa}\partial_r\xi)\,\ddot{z}^a\right](\dot{z}^e\partial_e\xi)^{-2} + \\ + (\partial_q\mathfrak{u}_{ra} - \partial_r\mathfrak{u}_{qa} + \mathfrak{b}_{ra}\partial_q\xi - \mathfrak{b}_{qa}\partial_r\xi)\,\dot{z}^a(\partial_b\xi\,\dot{z}^b)^{-1}\Big\}_{\tau=\tau^{\pm}} \mp \\ \mp\, e\int_{\tau\pm}^{\pm\infty}\mathfrak{f}_{qra}\,\dot{z}^a\,d\tau + O(\xi)\,, \tag{3.47}$$

where

$$\mathfrak{f}_{qra} = (\partial_q\mathfrak{b}_{ra} - \partial_r\mathfrak{b}_{qa})\,. \tag{3.48}$$

§3.2.2. Global damping

Energy-momentum tensor density. We consider the energy-momentum tensor density $\Im^{ab} = (\Im^{ab})_{\text{mass}} + (\Im^{ab})_{\text{field}} + (\Im^{ab})_{\text{elec}}$ (3.37) of the system at the point x. Thus (here O vanishes when $\xi\to 0$),

$$\frac{1}{c}\,\mathfrak{t}^a_q\,\Im^{qr}dS_r = \frac{1}{4\pi c}\sqrt{-g}\,\Big[\,\mathfrak{t}^a_q\big(\langle F^{q\cdot}_{\cdot s}\rangle\langle F^{rs}\rangle + \langle (F^{q\cdot}_{\cdot s})^{\text{free}}\rangle\langle F^{rs}\rangle + \\ + \langle F^{q\cdot}_{\cdot s}\rangle\langle (F^{rs})^{\text{free}}\rangle\big)\,dS_r - \left(\frac{1}{4}\langle F_{st}\rangle\langle F^{st}\rangle + \frac{1}{2}\langle (F_{st})^{\text{free}}\rangle\langle F^{st}\rangle\right)\times \\ \times\,\mathfrak{t}^{qa}\,dS_q\Big] + \frac{1}{c}\,\mathfrak{t}^a_q\,(\Im^{qr})_{\text{field}} + O(\xi)\,. \tag{3.49}$$

Avoiding the mass renormalization. Reverting to the result inferred in the present theory, we integrate (3.49) according to

$$\frac{1}{c}\int_{4\pi}\mathfrak{t}^a_q\,\Im^{qr}dS_r = \left[\frac{e^2}{2\epsilon c^2}\,\ddot{z}^a - \frac{e^2}{c}\,\dot{z}^b\int_{-\infty}^{+\infty}\mathfrak{f}^{a\cdot\cdot}_{\cdot be'}\,\dot{z}^{e'}\tau'\,d\tau' - \\ - \frac{e}{c}\,\langle (F^{a\cdot}_{\cdot b})^{\text{free}}\rangle\,\dot{z}^b\right]d\tau + \frac{1}{c}\int_{4\pi}\mathfrak{t}^a_q\,(\Im^{qr})_{\text{field}}\,dS_r + O(\xi)\,, \tag{3.50}$$

where $\ddot{z}^a$ is a function of the electromagnetic field.

In this equation, we must get rid of the term $\frac{e^2}{2\xi c^2}\,\ddot{z}^a$ which is divergent. This term has the same kinematical structure as the mass term in (2.58). Therefore, we renormalize the mass as follows

$$m = m_0 + \lim_{\xi\to 0}\frac{e^2}{2\,\xi\, c^2} \tag{3.51}$$

and (2.58) reads now

$$m\,\ddot{z}^a = \frac{e}{c}\,\langle (F^{a\cdot}_{\cdot b})^{\text{free}}\rangle\,\dot{z}^b + \frac{e^2}{2\,c}\,\dot{z}^b\int_{-\infty}^{+\infty}\mathfrak{f}^{a\cdot\cdot}_{\cdot be'}\,\dot{z}^{e'}\,d\tau'. \tag{3.52}$$

By setting

$$\ddot{z}^a\,\frac{e^2}{2\,\xi\,c^2}\,d\tau = -\frac{1}{c}\int_{4\pi}\mathfrak{t}^a_q\,(\Im^{qr})_{\text{field}}\,dS_r\,, \tag{3.53}$$

we remark that the renormalization is no longer required, which gives a better physical consistency to the present theory. This particular circumstance tends to lend support to the existence of free gravitational fields predicted by the EGR theory.

In the absence of charge, we obtain the well-known inertia law

$$m_0\,\ddot{z}^a = 0\,. \tag{3.54}$$

As outlined by De Witt and Brehme [3], for purposes of application to physically set boundary conditions, it is more appropriate to deal with the "incoming" field $(F_{ab})^{\text{in}}$

$$m_0\,\ddot{z}^a = \frac{e}{c}\,(F^{a\cdot}_{\cdot b})^{\text{in}}\,\dot{z}^b + \frac{2}{3}\,\frac{e^2}{c^3}\left(\dddot{z}^a - \frac{\dot{z}^a\,\ddot{z}^2}{c^2}\right) + \\ + \frac{e^2}{c}\,\dot{z}^b\int_{-\infty}^{r}\mathfrak{f}^{a\cdot\cdot}_{\cdot be'}\,\dot{z}^{e'}\,d\tau\,. \tag{3.55}$$

On the right hand side, one recognizes the first two terms of the relativistic equation (1.23), bearing in mind that the derivatives are covariant here, while keeping the proper mass m_0 on the left hand side.

The third term determined by $\mathfrak{b}_{aq}$ of the Green function is the "tail" due to the space-time curvature and radiation damping occurs even when $(F^{a\cdot}_{\cdot b})^{\text{in}}$ vanishes.

Concluding remarks

Let us stress some important points about the "tail" term:

- It is spanned by R_{ab}, which are built with the general connection coefficients (2.2): $\Gamma = \{\ \} + (\Gamma)_J$ as defined by the EGR theory;

- We thus have implicitly assumed that the elementary solutions of Hadamard and subsequent relations hold within the suggested extension of the GR theory.

Upon this assumption, it is clearly shown that, with the introduction of the related persistent free field, one no longer requires a negative external mass, thus avoiding an unphysical "pathology" found in the Riemannian theory.

In the Euclidean approximation, the third term (3.55) vanishes anyway and the formula (1.23) is recovered.

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