The EGR Theory: An Extended Formulation of General Relativity

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Abstract: This is an extended formulation of General Relativity based on the existence of an additional segment curvature, due to the non-vanishing covariant derivative of the metric tensor. The resulting enlarged manifold allows for a permanent "free" field to exist next to the usual phenomenological energy-momentum tensor. This field may provide plausible explanation to further unanswered pending issues.

Contents:

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Notations:

To completely appreciate this article, it is imperative to define some notations employed.

INDICES. Throughout this paper, we adopt the Einstein summation convention whereby a repeated index implies summation over all values of this index:

4-tensor or 4-vector: small Latin indices $a, b, \ldots = 1, 2, 3, 4$;

3-tensor or 3-vector: small Greek indices $\alpha, \beta, \ldots = 1, 2, 3$;

4-volume element: d^4x;

3-volume element: d^3x.

SIGNATURE OF SPACE-TIME METRIC:

Hyperbolic $(+---)$ unless otherwise specified.

OPERATIONS:

Scalar function: $U(x^a)$;

Ordinary derivative: $\partial_a U$;

Covariant derivative in GR: ∇_a;

Covariant derivative in EGR: D_a or $'$, (alternatively).

TENSORS:

Symmetrization: $A_{(ab)} = \frac{1}{2}\,!\,(A_{ab} + A_{ba})$;

Anti-symmetrization: $A_{[ab]} = \frac{1}{2}\,!\,(A_{ab} - A_{ba})$;

Kronecker symbol: $\delta_{ab} = (+1 \text{ if } a = b; \; 0 \text{ if } a \neq b)$;

Levi-Civita tensor: ϵ_{abcd} (where $\epsilon^{1234} = 0$).

THREE-DIMENSIONAL VECTORIAL QUANTITIES:

$\boldsymbol{P} = P_\alpha$.

Dedication:

The author dedicates this paper to the memory
of André Lichnérowicz (1915–1998).

As a young student in Theoretical Physics at the Faculty of Sciences-Jussieu-Paris, I enthusiastically began to follow the lectures on GR analysis given by Prof. André Lichnérowicz at the prestigious Collège de France where he held Chaire of Physique Mathématique since 1952.

He was himself a student of Georges Darmois, a pioneer in differential geometry, as well as Elie Cartan, gently nicknamed "Papa Cartan" because of his permanent availability to his students.

"Lichné" had certainly inherited those virtues: intelligent, broad-minded, curious and always accessible to "fresh men" as well as to post doctorate students. He generously arranged for me to have permanent access to the restricted library of the Institut Henri Poincaré, rue Curie, where I spent daylong research readings. This is indeed the evidence of a great humanist and a far reaching thinker internationally acknowledged.

It was then this autumn of 1985, when, already retired, he received me in his apartments, rue Paul Appel in South Paris, where we discussed the current status of Physics, although he had rather wished to remain a mathematician. (P. A. M. Dirac was in his opinion the greatest mind next to Einstein.)

In my last visit to him in 1995, I showed him a collection of my paper drafts including a new theory on a possible extension of General Relativity: he warmly encouraged me to proceed and kindly gave me several of his own text books, which I now modestly consider as a legacy.

As a dedication to his memory, I am honored to present here this paper.

Patrick Marquet

Introduction

As early as 1915, Einstein's General Theory of Relativity (GR) has successfully generalised Newton's original equations wherefrom most of the cosmological observations have been accurately described (to a certain extent).

As a possible doorway to further analysis, I would like to present here a new approach of the concept of gravity by considering a "free" gravity-like field which is assumed to be present and "localizable" throughout our Universe. Like the usual gravitational field classically resulting from the mass, this specific field interacts with matter and this coupling actually accounts for the known gravitational mass.

In this paper, our basic idea rests upon following observation. In the framework of classical physics, electrodynamics is described by means of two tensors:

- A pure electromagnetic field tensor described by Maxwell's tensor;
- A massive tensor which constitutes the charged particle.

Interaction of both quantities results in a conserved global momentum vector. Proceeding, in perfect analogy with the above, we suggest that gravitation also be described by two tensors:

- One tensor inherent to a pure field;
- The other tensor generalized only relative to the particle's mass.

By doing so however, we come across a major difficulty. Classical electrodynamics takes place in either an Euclidean space or on a Riemannian manifold. A straightforward gravitational analogy is not admissible, for whatever be the gravity field, it defines the space-time structure, which in turn will affect the matter field coupling.

Our line of attack consists of assigning to the macroscopic energy-momentum tensors a "dominant Riemannian" characteristic which is embedded in a more global geometry. In the framework of this scheme, the Riemannian physics would then just appear as a large scale approximation characterizing the elementary masses and energies, thus never conflicting with the known results of GR. On the very small scale, however, the non-Riemannian geometry is no longer negligible and its properties should be taken into consideration.

To achieve such a construction, we develop an antisymmetric extended (torsion-free) General Relativity (while keeping the four space-time dimensions), by ruling out the restrictive metric condition $\nabla g_{ab}=0$, thus introducing a new connection built from the non-vanishing covariant derivative of the metric tensor.

The resulting enlarged manifold displays here an extra curvature called the *segment curvature*.

With this preparation, we can derive a generalized Einstein tensor denoted here the *EGR tensor* (i.e. the Extended GR tensor), which implies the existence of the so-called *EGR field equations* very close to the classical case.

In the absence of energy (e.g. mass), the EGR field equations however, do not reduce to the Riemannian source free equations: they actually always retain a "remnant-like" energy-momentum field tensor, which can be regarded as a vacuum "background" displaying a non-vanishing low level Riemannian part and non-Riemannian part due to the covariant derivative of the metric tensor.

As a conceptual gift, with this new theory, one no longer requires a "vanishing" (symmetric) gravitational energy-momentum pseudo-tensor [1] attributed to the mass and whose physical meaning has always remained unclear. In this sense, the EGR "residual" (true) field tensor is just a continuation of this pseudo-tensor when escaping a massive body. As well, its deep antisymmetric nature arises naturally from the theory.

It clearly confirms Einstein's early choice (as well as Dirac), and thus simply avoids the confusing controversy between the two versions. Last but not the least, the cosmological constant term $g_{ab}\lambda$, which is initially discarded in the text, automatically reappears under the form of a (small) term $g_{ab}J^2$ where J^2 is the square of a slightly varying four-vector fundamentally related to the extra segment curvature.

In our opinion, J^2, which prevails among other terms on the right hand side of the EGR field equations, has been "erroneously" approximated to the famous constant λ, thus misleading, since the complete structure of the EGR equations has been ignored.

Chapter 1. Gravitational Field: The Classical Theory

§1.1. The GR fundamental equations

Typically, the source-free field equations are non-linear equations of propagation which must contain derivatives of g_{ab} up to 2.

So, we consider the action

$$S = \int L_{\mathrm{E}}\sqrt{-g}\; d^4x\,, \qquad \det\|\,g_{ab}\| = g\,, \tag{1.1}$$

which must be stationary when the metric tensor is varied and where the Lagrangian L_{E} and its density $\mathcal{L}_{\mathrm{E}}$ are expressed with the Christoffel symbols as in the classical Einsteinian theory (Riemannian geometry)

$$\mathcal{L}_{\rm E} = \sqrt{-g}\, L_{\rm E} = \sqrt{-g}\, g^{ab} \left(\left\{ {e \atop ab} \right\} \left\{ {d \atop de} \right\} + \left\{ {d \atop ae} \right\} \left\{ {e \atop bd} \right\} \right) \tag{1.2}$$

being derived from the contracted curvature tensor (Ricci's tensor)

$$R_{bc} = \partial_a \left\{ {a \atop bc} \right\} - \partial_c \left\{ {a \atop ba} \right\} + \left\{ {d \atop bc} \right\} \left\{ {a \atop da} \right\} - \left\{ {d \atop ba} \right\} \left\{ {a \atop dc} \right\} . \tag{1.3}$$

Thus one infers the source-free field equations

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab} R = 0\,. \tag{1.4}$$

The Einstein tensor G_{ab} is a symmetric second-rank tensor, which is a function of only g_{ab} and their first and second derivatives. We have thus ten equations in (1.4) with partial derivatives which are not mutually independent.

There exists only 6 independent conditions, since the space-time coordinates can be subjected to an arbitrary transformation allowing us to choose four out of the ten generalizations of the metric tensor g_{ab}.

In order for the four conservation identities resulting from (1.4)

$$\nabla_a G^a_b = 0 \tag{1.5}$$

to be satisfied along with the previous conditions, Elie Cartan showed that the tensor G_{ab} should have the following form

$$G_{ab} = k \left[R_{ab} - \frac{1}{2}\, g_{ab} \left(R - 2\lambda \right) \right], \qquad k = const, \tag{1.6}$$

where λ is known as the cosmological constant, and it will be discarded here. When a source (matter) is present, we obtain ten non-linear equations

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab} R = \varkappa T_{ab}\,, \tag{1.7}$$

which show that masses and space-time are not mutually independent. Also, here

$$\varkappa = -\frac{8\pi\mathfrak{G}}{c^4} \tag{1.8}$$

is Einstein's constant and $\mathfrak{G}$ is Newton's constant.

The (massive) energy-momentum tensor here is given by

$$T_{ab} = \rho c^2 u_a u_b\,, \tag{1.9}$$

where ρ is the matter density.

The fundamental equation (1.8) generalizes the Poisson equation, which is clearly valid in Newtonian physics when the macroscopic velocities are slow compared to the light velocity c.

§1.2. Energy-momentum pseudo-tensor density

The condition $\nabla_a G^a_b = 0$ implies

$$\nabla_a T^a_b = 0$$

or

$$\partial_a \Im^a_b = 0 \tag{1.10}$$

with the tensor density

$$\Im^a_b = \sqrt{-g}\; T^a_b \,.$$

However, inspection shows that

$$\partial_a \Im^a_b = \frac{1}{2}\, \Im^{cd}\, \partial_b\, g_{cd}$$

and the condition (1.10) is thus never satisfied in a general coordinate system.

The classical theory requires that the total four-momentum of matter and its gravitational field

$$P^a = \frac{1}{c} \int (T^{ab} + t^{ab}) \sqrt{-g}\; dS_b$$

should be conserved.

We thus have to introduce a tensor density

$$\mathcal{T}_{ab} = \sqrt{-g}\; t_{ab}$$

such that

$$\partial_a (\Im^a_b + \mathcal{T}^a_b) = 0 \tag{1.11}$$

with the explicit form

$$\mathcal{T}^c_d = \frac{1}{2\varkappa} \left[\frac{(\partial_d\, \mathcal{G}^{ab})\, \partial \mathcal{L}_{\text{E}}}{\partial\, (\partial_c\, \mathcal{G}^{ab})} - \delta^c_d\, \mathcal{L}_{\text{E}} \right], \tag{1.12}$$

where

$$\mathcal{G}^{ab} = \sqrt{-g}\; g^{ab}$$

is the *metric tensor density*, constructed from the fundamental metric tensor g_{ab}.

The quantities $\mathcal{T}^{ab}$ are called *pseudo-tensor densities* of Landau-Lifshitz, for they can be transformed away by a suitable choice of the reference frame. The densities $\mathcal{T}^{ab}$ are just formed with the Christoffel symbols, themselves becoming a generalization of a true tensor only with respect to linear coordinate transformations. This is why the classical theory stipulates that the gravitational energy, which is attributed to masses, is *not localizable* and therefore *cannot be engineered*.

Chapter 2. The Basics of the EGR Theory

§2.1. Extended Riemannian geometry

§2.1.1. Structure of the extended manifold

Consider the generalization $\mathsf{R}^{e\cdot\cdot\cdot}_{\cdot acd}$ having the same form as the Riemann curvature tensor $R^{e\cdot\cdot\cdot}_{\cdot acd}$, but constructed on other connection coefficients

$$\mathsf{R}^{e\cdot\cdot\cdot}_{\cdot acd} = \partial_d \Gamma^e_{ac} - \partial_c \Gamma^e_{ad} + \Gamma^e_{ac}\Gamma^k_{kd} - \Gamma^k_{ad}\Gamma^e_{kc}\,. \tag{2.1}$$

On a manifold M referred to a natural basis, e_a, it is known that the connection coefficients Γ^c_{ab} can be decomposed as follows

$$\Gamma^c_{ab} = \{^{\,c}_{ab}\} + K^c_{ab} + (\Gamma^c_{ab})_s\,, \tag{2.2}$$

where $\{^{\,c}_{ab}\}$ are the conventional Christoffel symbols of the second kind, used in General Relativity, and (see Tonnelat [1, p. 30–32] for detail)

$$K^c_{ab} = \frac{1}{2}\, g^{ce}\left(T_{[ae],b} + T_{[be],a} + T_{[ab],e}\right) \tag{2.3}$$

is referred to as the *contorsion tensor*, which includes the torsion tensor* $T^c_{[ba]} = \frac{1}{2}(\Gamma^c_{ba} - \Gamma^c_{ab})$. The quantity

$$(\Gamma^c_{ab})_s = \frac{1}{2}\, g^{ce}\left(\mathrm{D}_b\, g_{ae} + \mathrm{D}_a\, g_{be} - \mathrm{D}_e\, g_{ab}\right) \tag{2.4}$$

is the so-called *segment connection*, which is formed with the covariant derivatives of the metric tensor

$$\mathrm{D}_c\, g_{ab} = \partial_c\, g_{ab} - \Gamma_{ac,b} - \Gamma_{bc,a}\,. \tag{2.5}$$

This last connection characterizes a particular property of the manifold M, which is related to a specific type of curvature called the "segment curvature".

In a dual basis θ defined on M, to any parallel-transported vector along a closed path can be associated:

- A rotation curvature

$$\Omega^a_b = -\frac{1}{2}\,\mathsf{R}^{a\cdot\cdot\cdot}_{\cdot bcd}\,\theta^c \wedge \theta^d; \tag{2.6}$$

- A torsion

$$\Omega^a = \frac{1}{2}\, T^a_{cd}\,\theta^c \wedge \theta^d; \tag{2.7}$$

- A segment curvature

$$\Omega = -\frac{1}{2}\,\mathsf{R}^{a\cdot\cdot\cdot}_{\cdot acd}\,\theta^c \wedge \theta^d. \tag{2.8}$$

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*Somewhere in the scientific literature, the torsion tensor is used in the other form $T^c_{[ba]} = \Gamma^c_{ba} - \Gamma^c_{ab}$ that does not matter in the present case.

§2.1.2. Modified action principle

As we know, the classical General Relativity is constructed from the Riemannian action $S = \int L_{\rm E} \sqrt{-g}\, d^4x$ (1.1), which is varied with respect to g_{ab}. The derived source free equations are

$$G_{ab} = R_{ab} - \frac{1}{2}\, g_{ab} R = 0\,.$$

In this case, the geometry is Riemannian, i.e.

$$T^c_{[ab]} = 0\,,$$

$$\mathrm{D}\, g_{ab} = \nabla g_{ab} = 0\,.$$

At this stage, our way of generalizing the GR theory is legitimized by the following remarks:

- The symmetry of the Einstein tensor is not sufficiently natural. Indeed, when derived from the relativistic theory, the canonical energy-momentum tensor is always antisymmetric, as in the electromagnetic field with a source

  $$\theta^{bc} = \frac{1}{4}\, g^{bc} F_{de} F^{de} - F^{ba} \partial^c A_a + g^{bc} j_a A^a$$

  and, in order to fit in the field equations, this tensor has to afterwards be symmetrized;
- The condition $\mathrm{D}\, g_{ab} \neq 0$ is more general than the restrictive Riemannian condition

  $$\nabla g_{ab} = 0\,;$$
- Moreover, we deem that the torsion tensor T^c_{ab} resorts more to an artificial mathematical property and does not offer a full physical and clear meaning. We therefore postulate a torsion-free manifold with 40 general symmetrical connection coefficients

  $$\Gamma^a_{bc} = \left\{{a \atop bc}\right\} + (\Gamma^a_{bc})_J\,, \qquad (2.9)$$

  where the latter connection is not necessary (2.4).

The Riemannian manifold should nevertheless be recovered when $\mathrm{D}\, g_{ab} = 0$. To begin with, we follow here the basic ideas of Einstein: instead of the potentials g_{ab}, we consider 40 connection coefficients (2.9) as the "field" variables.

In this context, the generalization of the Ricci tensor formed with Γ^a_{bc} is still expressed by

$$R_{bc} = \partial_a \Gamma^a_{bc} - \partial_c \Gamma^a_{ba} + \Gamma^d_{bc} \Gamma^a_{da} - \Gamma^d_{ba} \Gamma^a_{dc}\,. \qquad (2.10)$$

§2.1.3. Eulerian equations

We consider the tensor density

$$\mathfrak{R}^{ab} = R^{ab}\sqrt{-g}\ , \tag{2.11}$$

from which we construct the invariant density

$$\mathcal{H} = \mathfrak{R}^{ab} R_{ab} \tag{2.12}$$

with

$$\mathfrak{R}^{ab} = \frac{\partial \mathcal{H}}{\partial R_{ab}}\,.$$

The least action principle is then

$$\delta S = \int \delta\mathcal{H}\, d^4x = 0\,. \tag{2.13}$$

For a variation $\delta\Gamma^a_{bc}$, we obtain

$$\delta S = \int \left[\left(\frac{\partial\mathcal{H}}{\partial\Gamma^a_{bc}}\right)\delta\Gamma^a_{bc} + \left(\frac{\partial\mathcal{H}}{\partial(\partial_e\,\Gamma^a_{bc})}\right)\delta\left(\partial_e\,\Gamma^a_{bc}\right)\right] d^4x = 0\,. \tag{2.14}$$

The variation of $\mathcal{H}$ is also expressed by

$$\delta\int \mathfrak{R}^{bc} R_{bc}\, d^4x = \int\left[\frac{\mathfrak{R}^{bc}\partial R_{bc}\,\delta\Gamma^a_{de}}{\partial\Gamma^a_{de}} + \frac{\mathfrak{R}^{bc}\partial R_{bc}\,\delta(\partial_k\Gamma^a_{de})}{\partial\left(\partial_k\Gamma^a_{de}\right)}\right] d^4x$$

and, integrating by parts, we obtain

$$\delta\int\left[\frac{\mathfrak{R}^{bc}\,\partial R_{bc}}{\partial\Gamma^a_{de}} - \partial_k\left(\frac{\mathfrak{R}^{bc}\,\partial R_{bc}}{\partial(\partial_k\Gamma^a_{de})}\right)\right]\delta\Gamma^a_{de} + \\ + \int\partial_k\left[\mathfrak{R}^{bc}\,\frac{\partial R_{bc}\,\delta\Gamma^a_{de}}{\partial(\partial_k\Gamma^a_{de})}\right] d^4x = 0\,. \tag{2.15}$$

If the variations $\delta\Gamma^a_{de}$ are zero on the integration boundary, the last divergence integral has no contribution.

The condition (2.15) reduces to

$$\delta\int\mathfrak{R}^{ab} R_{ab}\, d^4x = \int\left(Q^{bc}_a\,\delta\Gamma^a_{bc}\right) d^4x = 0 \tag{2.16}$$

with

$$Q^{bc}_a = \mathfrak{R}^{de}\,\frac{\partial R_{de}}{\partial\Gamma^a_{bc}} - \partial_k\left[\mathfrak{R}^{de}\,\frac{\partial R_{de}}{\partial\left(\partial_k\Gamma^a_{bc}\right)}\right]. \tag{2.17}$$

The stationary principle for the symmetric Γ^a_{bc} leads to the Eulerian equations

$$Q^{(bc)}_a = \frac{1}{2}\left(Q^{bc}_a + Q^{cb}_a\right) = 0\,. \tag{2.18}$$

From the expression (2.10), we derive the derivatives

$$\frac{\partial R_{dk}}{\partial\left(\partial_e \Gamma^a_{bc}\right)} = \delta^e_m \delta^b_d \delta^c_k \delta^m_a - \delta^e_k \delta^b_d \delta^c_m \delta^m_a \tag{2.19}$$

and

$$\frac{\partial R_{dk}}{\partial \Gamma^a_{bc}} = \delta^n_a \delta^b_d \delta^c_k \Gamma^m_{nm} + \delta^m_a \delta^b_n \delta^c_k \Gamma^n_{dm} - \\ - \delta^n_a \delta^b_d \delta^c_m \Gamma^m_{nk} - \delta^m_a \delta^b_n \delta^c_k \Gamma^n_{dm}\,. \tag{2.20}$$

Now substituting these into (2.17) yields

$$-\,Q^{bc}_a = \partial_a \mathfrak{R}^{bc} - \delta^c_a \partial_e \mathfrak{R}^{be} - \mathfrak{R}^{bc}\,\Gamma^m_{am} - \delta^k_a \mathfrak{R}^{dc}\,\Gamma^b_{dk} + \\ + \mathfrak{R}^{bk}\,\Gamma^c_{ak} + \mathfrak{R}^{kc}\,\Gamma^b_{ka} = \left(\mathfrak{R}^{bc}\right)_{\prime,a} - \delta^c_a \left(\mathfrak{R}^{be}\right)_{\prime,e} \tag{2.21}$$

with

$$\left(\mathfrak{R}^{bc}\right)_{\prime,a} = \partial_a \mathfrak{R}^{bc} + \Gamma^b_{ea}\,\mathfrak{R}^{ec} + \Gamma^c_{ea}\,\mathfrak{R}^{eb} - \Gamma^e_{ae}\,\mathfrak{R}^{bc} \tag{2.22}$$

where $\prime,$ are the covariant derivatives constructed with the global Γ^a_{bc} defined in (2.9).

The condition (2.18) explicitly yields

$$\left(\mathfrak{R}^{bc} + \mathfrak{R}^{cb}\right)_{\prime,a} - \delta^c_a \left(\mathfrak{R}^{be}\right)_{\prime,e} - \delta^b_a \left(\mathfrak{R}^{ce}\right)_{\prime,e} = 0\,. \tag{2.23}$$

§2.2. Connection coefficients

In order to determine the exact form of the connection, we first decompose $\mathfrak{R}^{bc}$ into the metric density $\mathcal{G}^{bc} = \sqrt{-g}\,g^{bc}$ and two parts $\mathcal{E}^{bc} + \mathcal{A}^{bc}$, where $\mathcal{A}^{bc}$ is antisymmetric

$$\mathfrak{R}^{bc} = \left(\mathcal{G}^{bc} + \mathcal{E}^{bc}\right) + \mathcal{A}^{bc}. \tag{2.24}$$

The two-term quantity in brackets represents the Riemann-Ricci tensor density

$$R^{bc}\sqrt{-g} = \mathcal{R}^{bc} = \mathcal{G}^{bc} + \mathcal{E}^{bc} \tag{2.25}$$

so that when $\mathcal{A}^{bc} = 0$, (2.24) reduces, as it should be, to (2.25).

Consistency of our theory leads to impose the following constraint

$$\left(\mathcal{E}^{bc}\right)_{\prime,b} = 0\,. \tag{2.26}$$

So forth we set

$$\mathcal{J}^b = \left(\mathcal{A}^{ba}\right)_{\prime,a} = \partial_a \mathcal{A}^{ba} \tag{2.27}$$

(due to the antisymmetry of $\mathcal{A}^{ab}$) with

$$\mathcal{J}^a = \sqrt{-g}\;J^a,$$

where the four-vector J^a will play a central role.

We now aim to check whether the condition $(\mathcal{G}^{bc})_{',c} = 0$ reinstates a Riemannian connection whereby the curvature tensor $\mathcal{R}_{ab}$ (2.10) would reduce to the Riemann-Ricci tensor R_{ab}.

By contracting (2.23) on c and a, and taking into account (2.26), one finds

$$(\mathcal{G}^{bc})_{',a} = -\frac{5}{3}\,\mathcal{J}^b. \tag{2.28}$$

If inserting (2.28) into (2.23), the conditions (2.18) eventually read

$$(\mathcal{G}^{bc})_{',a} = -\frac{1}{3}\left(\delta^b_a\,\mathcal{J}^c + \delta^c_a\,\mathcal{J}^b\right). \tag{2.29}$$

Dividing by $\sqrt{-g}$, we obtain

$$\partial_a g^{bc} + g^{bc}\partial_a \ln\sqrt{-g} + \Gamma^b_{ea}\,g^{ec} + \Gamma^c_{ea}\,g^{be} - \Gamma^e_{ea}\,g^{bc} = = -\frac{1}{3}\left(\delta^b_a\,J^c + \delta^c_a\,J^b\right) \tag{2.30}$$

and multiplying through by g_{bc}, having $g_{ba}\,g^{ca} = \delta^c_b$ taken into account as well as

$$dg = g\,g^{bc}\,dg_{bc} = -\,g\,g_{bc}\,dg^{bc},$$

we infer

$$\Gamma^e_{ae} = \partial_a \ln\sqrt{-g} + \frac{1}{3}\,J_a\,. \tag{2.31}$$

Substituting this last relation into (2.30) and multiplying it by $g_{bd}\,g_{kc}$ (after noting that $dg_{ed} = -\,g_{ec}\,g_{bd}\,dg^{bc}$), we eventually find

$$\partial_a g_{bc} - \Gamma^k_{ba}\,g_{kc} - \Gamma^k_{ca}\,g_{bk} = \frac{1}{3}\left(J_c\,g_{ab} + J_b\,g_{ac} - J_a\,g_{bc}\right) = \mathrm{D}_a\,g_{bc}\,. \tag{2.32}$$

Interchanging the indices a and b, then a and c, we obtain two more equations of type (2.32), which could be virtually denoted by $(2.32)'$ and $(2.32)''$. From the linear combination $(2.32)' + (2.32)'' - (2.32)$, we eventually get the explicit form of the global connection

$$\Gamma^d_{ab} = \left\{{d \atop ab}\right\} + (\Gamma^d_{ab})_J = \left\{{d \atop ab}\right\} + \frac{1}{6}\left(\delta^d_a\,J_b + \delta^d_b\,J_a - 3g_{ab}\,J^d\right). \tag{2.33}$$

Our last equation (2.33) shows that when $J_a = 0$, we have $\mathrm{D}_a\,g_{bc} = 0$ and thus

$$(\mathcal{G}^{ab})_{',b} = 0\,. \tag{2.34}$$

From (2.31), the condition $J_a = 0$ implies $(\Gamma^b_{ae})_J = 0$, so we see that in the case, the generalized curvature tensor $\mathcal{R}_{ab}$ (2.10) reduces to the Riemann-Ricci tensor R_{ab}.

Chapter 3. The EGR Field Equations

§3.1. EGR curvature tensors

§3.1.1. The fourth-rank curvature tensor

From the connection

$$\Gamma^d_{ab} = \left\{{}^{\,d}_{ab}\right\} + (\Gamma^d_{ab})_J = \left\{{}^{\,d}_{ab}\right\} + \frac{1}{6}\left(\delta^d_a J_b + \delta^d_b J_a - 3\, g_{ab} J^d\right) \tag{3.1}$$

the EGR curvature tensor can be derived

$$\mathsf{R}^{a\cdots}_{\cdot\, bcd} = R^{a\cdots}_{\cdot\, bcd} + \nabla_d \Gamma^a_{bc} - \nabla_c \Gamma^a_{bd} + \Gamma^k_{bc}\Gamma^a_{kd} - \Gamma^k_{bd}\Gamma^a_{kc}\,. \tag{3.2}$$

Inspection shows that the following relations hold

$$(\mathsf{R}^{e\cdots}_{\cdot\, dab})_{\prime,\,k} + (\mathsf{R}^{e\cdots}_{\cdot\, dka})_{\prime,b} + (\mathsf{R}^{e\cdots}_{\cdot\, dbk})_{\prime,\,a} = 0\,, \tag{3.3}$$

$$\mathsf{R}^{e\cdots}_{\cdot\, dab} + \mathsf{R}^{e\cdots}_{\cdot\, bda} + \mathsf{R}^{e\cdots}_{\cdot\, abd} = 0\,. \tag{3.4}$$

Let us now contract

$$g_{ce}\,\mathsf{R}^{e\cdots}_{\cdot\, dab} = \mathsf{R}_{cdab}\,, \tag{3.5}$$

we then note that, from $\nabla_a(\Gamma^e_{db})_J$,

$$g_{ce}\nabla_a\big[(\Gamma^e_{bk})_J\,\delta^k_m\,\delta^m_d\big] = g_{cd}\nabla_a(\Gamma^e_{be})_J$$

and the curvature tensor (3.5) now reads

$$\mathsf{R}_{cdab} = R_{cdab} + g_{ce}\nabla_b(\Gamma^e_{da})_J - \frac{1}{2}\, g_{ce}\big[\nabla_a(\Gamma^e_{db})_J + \nabla_d(\Gamma^e_{ab})_J\big] + $$
$$+ g_{ce}\big[(\Gamma^e_{kb})_J(\Gamma^k_{da})_J - (\Gamma^e_{ka})_J(\Gamma^k_{db})_J\big] + g_{cd}\big[\partial_a(\Gamma^e_{be})_J - \partial_b(\Gamma^e_{ae})_J\big]. \tag{3.6}$$

With the definition (3.1) we have

$$(\Gamma^d_{ad})_J = \frac{1}{3}\, J_a \tag{3.7}$$

and

$$\partial_a(\Gamma^d_{bd})_J - \partial_b(\Gamma^d_{ad})_J = \frac{1}{3}\, J_{ab} \tag{3.8}$$

with

$$J_{ab} = \partial_a J_b - \partial_b J_a\,. \tag{3.9}$$

§3.1.2. The EGR second-rank tensor

The relation (3.6) eventually leads to the contracted tensor

$$\mathsf{R}^{d\cdots}_{\cdot\, abd} = \mathsf{R}_{ab} = R_{ab} + \nabla_d(\Gamma^d_{ab})_J - \nabla_b(\Gamma^d_{ad})_J + $$
$$+ (\Gamma^k_{ab})_J(\Gamma^d_{kd})_J - (\Gamma^k_{ae})_J(\Gamma^e_{kb})_J\,, \tag{3.10}$$

we then have once more the splitting

$$\mathsf{R}_{ab} = \mathsf{R}_{(ab)} + \mathsf{R}_{[ab]} \tag{3.11}$$

with

$$\mathsf{R}_{(ab)} = R_{ab} + \nabla_d (\Gamma^d_{ab})_J - \frac{1}{2}\left[\nabla_b (\Gamma^d_{ad})_J + \nabla_a (\Gamma^d_{bd})_J\right] + \\ + (\Gamma^k_{ab})_J (\Gamma^d_{kd})_J - (\Gamma^k_{ae})_J (\Gamma^e_{kb})_J \tag{3.12}$$

and

$$\mathsf{R}_{[ab]} = \frac{1}{2}\left[\partial_a (\Gamma^d_{bd})_J - \partial_b (\Gamma^d_{ad})_J\right] \tag{3.13}$$

that is

$$\mathsf{R}_{(ab)} = R_{ab} - \frac{1}{2}\left(g_{ab} \nabla_d J^d + \frac{1}{3} J_a J_b\right), \tag{3.14}$$

$$\mathsf{R}_{[ab]} = \frac{1}{6} (\partial_a J_b - \partial_b J_a). \tag{3.15}$$

§3.1.3. The EGR curvature scalar

Applying $\mathsf{R} = g^{da} \mathsf{R}_{da}$, we have

$$\mathsf{R} = R - \nabla_e \left[g^{da} (\Gamma^e_{da})_J\right] - \nabla_e \left[g^{dc} (\Gamma^e_{dc})_J\right] - \\ - g^{da}\left[(\Gamma^e_{da})_J (\Gamma^c_{ce})_J - (\Gamma^k_{de})_J (\Gamma^e_{ka})_J\right] \tag{3.16}$$

or

$$\mathsf{R} = R - \frac{1}{3}\left(\nabla_e J^e + \frac{1}{2} J^2\right). \tag{3.17}$$

§3.2. The EGR Einstein tensor

Unlike the Riemann curvature tensor, the EGR curvature tensor is no longer antisymmetric on the indices pair ca

$$\mathsf{R}_{cabk} + \mathsf{R}_{acbk} = \frac{2}{3} g_{ca} J_{bk} \tag{3.18}$$

or

$$\mathsf{R}^{ca\cdot\cdot}_{\cdot\cdot bk} + \mathsf{R}^{ac\cdot\cdot}_{\cdot\cdot bk} = \frac{2}{3} g^{ca} J_{bk}\,. \tag{3.19}$$

Lifting the indices d in the equation (3.3) and contracting on d and k as well as on b and e, we obtain

$$(\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ab})_{\prime,k} + (\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ka})_{\prime,b} + (\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot bk})_{\prime,a} = 0 \tag{3.20}$$

then we replace $\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ab}$ by its value from (3.19). We eventually find

$$(\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot bk})_{\prime,a} + 2\,(\mathsf{R}^{bk\cdot\cdot}_{\cdot\cdot ab})_{\prime,k} + \frac{2}{3} g^{bk} (J_{ka})_{\prime,b} = 0 \tag{3.21}$$

that is

$$\left(\mathsf{R}_{a}^{k}-\frac{1}{2}\,\delta_{a}^{k}\,\mathsf{R}\right)_{\prime,k}=-\frac{1}{3}\,(J_{a}^{k})_{\prime,k} \tag{3.22}$$

which is just the generalized conservation law for the EGR tensor $\mathsf{G}_{da} = (G_{da})_{\mathrm{EGR}}$ (here we substitute $d = k$)

$$\mathsf{G}_{da}=\mathsf{R}_{(da)}-\frac{1}{2}\left(g_{da}\,\mathsf{R}-\frac{2}{3}\,J_{da}\right). \tag{3.23}$$

The latter will be called here the *EGR Einstein Tensor.* It obviously reduces to the "Riemannian" Einstein tensor

$$G_{da}=R_{da}-\frac{1}{2}\,g_{da}\,R=0$$

in the framework of the classical GR field equations.

The equations (3.23) are a transcription of the tensor density EGR field equations

$$\mathfrak{R}^{da}+\mathcal{B}^{da}=0\,, \tag{3.24}$$

whose conservation law is

$$\left(\mathfrak{R}_{a}^{b}+\mathcal{B}_{a}^{b}\right)_{\prime,b}=0\,.$$

In the strong (ideal) Riemannian regime $\mathcal{J}^{a}=0$, thus

$$(\mathfrak{R}_{a}^{b})_{\prime,b}=\nabla_{b}\,\mathcal{R}_{a}^{b}=0\,,$$

$$\partial_{b}\,\mathcal{R}_{a}^{b}-\left\{{}_{ba}^{\,c}\right\}\mathcal{R}_{c}^{b}=0\,,$$

or

$$\partial_{b}\,\mathcal{R}_{a}^{b}-\frac{1}{2}\,\mathcal{R}^{cb}\,\partial_{a}\,g_{cb}=0$$

eventually

$$\nabla_{b}\left(R_{a}^{b}-\frac{1}{2}\,\delta_{a}^{b}\,R\right)=0\,,$$

which is just the conserved Einstein tensor G_{ab} as inferred from the Bianchi identities in the classical GR

$$G_{ab}=R_{ab}-\frac{1}{2}\,g_{ab}\,R\,.$$

It is now easy to derive the expression of the tensor B_{da} corresponding to $\mathcal{B}^{da}$

$$B_{da}=-\frac{1}{2}\left(\frac{3}{2}\,g_{da}\,\nabla_{e}\,J^{e}+\frac{1}{3}\,J_{d}\,J_{a}-\frac{1}{6}\,g_{da}\,J^{2}+\frac{2}{3}\,J_{da}\right). \tag{3.25}$$

By doing so, we note that $\frac{1}{6}\,g_{da}\,J^{2}$ is only the term in the bracket which carries J^{2}, and which prevails over the others as a candidate to generalize $g_{da}\,\lambda$ (the cosmological term). The term $\frac{1}{6}\,g_{da}\,J^2$ is reminiscent of the classical $g_{da}\,\lambda$, where λ was long regarded as a mere constant in the usual Riemannian theories.

§3.3. The persistent field

§3.3.1. The EGR field equations

In the framework of the EGR theory, our universe is completely described by $\mathcal{G}_{ab} = (G_{ab})_{\rm EGR}$.

In the classical GR, the source-free field equations are

$$G_{ab} = 0\,, \tag{3.26}$$

but according to our basic postulate, the latter "Riemannian" equation is merely a particular case in the framework of the global EGR geometry. Therefore, in the absence of the macroscopic energy term, there should always remain a faint energy tensor described by the extra curvature.

The classical vacuum equations (3.26) should be replaced by the following EGR field equations

$$\mathfrak{R}^{ab} + \mathcal{B}^{ab} = \varkappa\,(\Im^{ab})_{\rm field}\,. \tag{3.27}$$

When matter or ponderomotive energy is present, we simply write

$$\Im^{ab} = (\Im^{ab})_{\rm Riem} + (\Im^{ab})_{\rm field}\,, \tag{3.28}$$

which has a certain analogy with the "Riemannian" electrodynamics, where there exists a massive tensor for a conductor, and an interacting electromagnetic energy-momentum tensor.

In the immediate neighbourhood of a mass, the Riemannian geometry represented by $(\Im^{ab})_{\rm Riem}$ becomes increasingly dominant inside the global one, and $(\Im^{ab})_{\rm field}$ coincides with the gravitational pseudo-tensor density classically attributed to the mass.

§3.3.2. The persistent energy-momentum tensor

By considering the tensor density $\mathcal{T}_{ab} = \sqrt{-g}\,t_{ab}$ (see Page 154),

$$\mathcal{T}^c_d = \frac{1}{2\varkappa}\left[(\partial_d\,\mathcal{G}^{ab})\,\frac{\partial\mathcal{L}_{\rm E}}{\partial\,(\partial_c\,\mathcal{G}^{ab})} - \delta^c_d\,\mathcal{L}_{\rm E}\right] \tag{3.29}$$

one can express the tensor density $(\Im^{ab})_{\rm field}$, which can be determined through the usual canonical equations

$$(\Im^{a}_{\cdot\,b})_{\rm field} = \frac{1}{2\varkappa}\left[\mathcal{H}\,\delta^a_b - (\partial_b\,\Gamma^e_{dk})\,\frac{\partial\mathcal{H}}{\partial\,(\partial_a\,\Gamma^e_{dk})}\right]. \tag{3.30}$$

It has a tensor counterpart $(T_{ab})_{\text{field}}$ which is written as

$$\sqrt{-g}\,(T^{ab})_{\text{field}} = (\Im^{ab})_{\text{field}}\,. \tag{3.31}$$

As expected from B_{ab}, we can easily check that this tensor is antisymmetric on the indices a and b.

In accordance with (3.28), we now suggest that the massive tensor density $(\Im^{ab})_{\text{Riem}}$ still be given by

$$(T_{ab})_{\text{Riem}} = \rho\, u_a u_b\,, \tag{3.32}$$

where ρ is the density of the (neutral) massive fluid.

The conservation law (1.10) then corresponds to

$$(\Im_{a\cdot}^{\cdot b})_{',b} = \left[(\Im_{a\cdot}^{\cdot b})_{\text{field}\,',b} + (\Im_{a\cdot}^{\cdot b})_{\text{Riem}\,',b}\right] = 0\,, \tag{3.33}$$

$$(\Im_{a\cdot}^{\cdot b})_{\text{mass}\,',b} = \partial_b(\Im_{a\cdot}^{\cdot b})_{\text{Riem}} - \left[\{^{\,c}_{ba}\}(\Im_{c\cdot}^{\cdot b})_{\text{Riem}} + (\Gamma^c_{ba})_J\,(\Im_{c\cdot}^{\cdot b})_{\text{Riem}}\right] = \\ = \rho\sqrt{-g}\,\frac{\mathrm{D}u_a}{d\tau}\,.$$

In the Riemannian regime, $\mathcal{J}^a = 0$ (which is an ideal case) and we should have

$$\nabla_b\,\Im_{a\cdot}^{\cdot b} = 0\,, \qquad \rho\sqrt{-g}\,\frac{\nabla u_a}{d\tau} = \partial_b\,(\Im_{a\cdot}^{\cdot b})_{\text{Riem}} - \{^{\,c}_{ba}\}\,(\Im_{c\,\cdot}^{\cdot b})_{\text{Riem}}$$

that is

$$\rho\,\frac{\nabla u_a}{d\tau} = \frac{1}{\sqrt{-g}}\,\partial_b\,(\Im_{a\cdot}^{\cdot b})_{\text{Riem}} - \{^{\,c}_{ba}\}\,(T^b_c)_{\text{Riem}} = \nabla_b\,T^b_a$$

in accordance with the classical result inferred from the definition of the massive tensor

$$T^{ab} = \rho u^a u^b.$$

Strictly speaking, the four-velocity u^a should be slightly modified since the Universe is characterized here by two forms:

- The quadratic form
$$ds^2 = g_{ab}\,dx^a dx^b;$$
- The linear form
$$dJ = f(J_b)\,dx^b.$$

A reasonable choice for $(u^a)_{\text{EGR}}$ can be

$$(u^a)_{\text{EGR}} = \frac{dx^a}{\sqrt{ds^2 + dJ}}\,. \tag{3.34}$$

§3.3.3. Interaction with matter

Reinstating the light velocity c, we now consider the immediate vicinity of a massive body. We write the total energy-momentum four-vector (attributed to field and mass)

$$P^a = \frac{1}{c}\int \left[(T^{ab})_{\text{field}} + (T^{ab})_{\text{Riem}}\right]\sqrt{-g}\; dS_b \tag{3.35}$$

across any given hypersurface.

In the framework of the immediate neighbourhood of the mass, the field tensor $(T_{ab})_{\text{field}}$ is replaced by the tensor t_{ab} which coincides with the classical gravitational energy-momentum pseudo-tensor.

In this case, the total energy-momentum four-vector reduces to the "Riemannian" result

$$P^a = \frac{1}{c}\int \left[\,t^{ab} + (T^{ab})_{\text{Riem}}\right]\sqrt{-g}\; dS_b\,. \tag{3.36}$$

Consider then the "contact" situation for which $(T_{ab})_{\text{Riem}}$, when integrated over the volume ϑ of the mass, gives the contribution

$$m_0c^2 = \int \left[(T^1_1)_{\text{Riem}} + (T^2_2)_{\text{Riem}} + (T^3_3)_{\text{Riem}} - (T^4_4)_{\text{Riem}}\right]\sqrt{-g}\; d\vartheta \tag{3.37}$$

into the total energy-momentum four-vector (3.35).

On the other hand, the "Riemannian" static field equations result in the follows

$$R^4_4 = \frac{8\pi\mathfrak{G}}{c^4}\left[(T^4_4)_{\text{Riem}} - \frac{1}{2}\,(T)_{\text{Riem}}\right] =$$

$$= \frac{4\pi\mathfrak{G}}{c^4}\left[(T^4_4)_{\text{Riem}} - (T^1_1)_{\text{Riem}} - (T^2_2)_{\text{Riem}} - (T^3_3)_{\text{Riem}}\right], \tag{3.38}$$

where we have first established that classically

$$\int R^4_4\sqrt{-g}\; d\vartheta = -\,4\pi\mathfrak{G}\,\frac{P^4}{c^3}\,, \qquad P^4 = m_0 c\,, \tag{3.39}$$

$$P^a = \frac{1}{c}\int \left[\,(\Im^{ab})_{\text{field}} + (\Im^{ab})_{\text{Riem}}\right] dS_b \tag{3.40}$$

across any given hypersurface.

At a large distance from a source, $(\Im^{ab})_{\text{Riem}} \to 0$, thus

$$P^a \approx \frac{1}{c}\int (\Im^{ab})_{\text{field}}\, dS_b\,. \tag{3.41}$$

Chapter 4. Concluding Remarks

As a temporary conclusion, we would like to consider the foregoing theoretical elements in the light of the long discussed "MOND" paradigm and its developments. Let us first recall some relevant history.

§4.1. The MOND formulation

Newtonian gravitational theory, when applied to describe acceleration of stars and gas as estimated from Doppler velocities, does not fit with the Newtonian field generated by the visible matter. This is known as the "missing mass" problem, which has led astrophysicists to invoke some sort of dark energy or exotic matter while it has actually never been detected.

In the meanwhile, some scientists have turned to a possible new law of gravity which would be more appropriate in predicting the observed anomalies. In the beginning of the 1980's, the astronomer Mordehai Milgrom [2] restated Newton's second law with the following scheme

$$\mu\left(\frac{a}{a_0}\right)\boldsymbol{a} = -\,\partial_\alpha \Phi_{\rm N} \tag{4.1}$$

where a is the generic acceleration, $\Phi_{\rm N}$ is the Newtonian potential of the visible matter, $\boldsymbol{a} = a_\alpha$ is the three-dimensional acceleration vector, and $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ is the three-dimensional spatial differential operator. Milgrom termed the acceleration scale a_0, where the function μ satisfies

$$\mu\,\frac{a}{a_0} = 1 \text{ for } a \gg a_0\,, \quad \text{and} \quad \mu\,\frac{a}{a_0} = \frac{a}{a_0} \text{ for } a \ll a_0 \tag{4.2}$$

with an estimate numerical value of

$$a_0 \approx 10^{-8}\,\text{cm/s}^2.$$

In the limit of low accelerations, Newton's second law should be quadratic and approach the following form (in the direction of the radial coordinate r) in the presence of a gravitational potential

$$\frac{a}{a_0}\,a_r = \partial_r \Phi\,.$$

For a point mass m the attractive potential at r is

$$V^4 = \mathfrak{G}\,m\,a_0\,,$$

which describes a flat rotation curve.

This is the *MOND paradigm*, the "modified Newtonian dynamics", which so far predicts most of the observed anomalies.

§4.2. Non-relativistic reformulation of MOND

A first Lagrangian (Riemannian) density was found to be

$$L = \frac{a_0^2}{8\pi\mathfrak{G}'}\, f\left(\frac{(\partial_\alpha\Phi)^2}{a_0^2}\right) - \rho\,\Phi\,, \tag{4.3}$$

where $(\partial_\alpha\Phi)^2 = \partial_\alpha\Phi\,\partial^\alpha\Phi$. This leads to the following gravitational field equation

$$\partial_\alpha\left[\mu\left(\frac{\sqrt{(\partial_\mu\Phi)^2}}{a_0}\right)\partial^\alpha\Phi\right] = 4\pi\mathfrak{G}'\rho\,, \tag{4.4}$$

where

$$\mu\left(\sqrt{y}\right) = \frac{df(y)}{dy}\,,$$

assuming

$$f(y) = \begin{cases} y & \text{for} \quad y \gg 1 \\ \frac{2}{3}\,y^{2/3} & \text{for} \quad y \ll 1 \end{cases}$$

and $\mathfrak{G}'$ is a constant which reduces to Newton's gravitational constant $\mathfrak{G}$ in the classical regime Φ_{N}. Inspection shows that, when the usual form of the generic acceleration is applied

$$\boldsymbol{a} = -\,\partial_\alpha\Phi\,, \tag{4.5}$$

the solution corresponds to (4.1).

The Lagrangian density (4.3) is "aquadratic", therefore the theory is known as the *AQUAL theory.*

§4.3. A theory of Te-Ve-S

First Relativistic AQUAL. It has been suggested to consider a physical metric $(g_{ab})'$ conformal to a "primitive" Einstein metric g_{ab} according to

$$(g_{ab})' = e^{2\Psi} g_{ab}\,, \tag{4.6}$$

where Ψ is a real scalar field.

The action of a particle of mass m_0 is expressed as

$$S = -\,m_0 \int e^{\Psi}\sqrt{-\,g_{ab}\,dx^a dx^b}\;. \tag{4.7}$$

For slow motion in a quasi-static situation with nearly flat metric and in a weak field Ψ,

$$e^{\Psi}\sqrt{-\,g_{ab}\,dx^a dx^b} \approx \left(1 + \Phi_{\text{N}} + \Psi - \frac{v^2}{2}\right) dt\,,$$

where

$$\Phi_{\rm N} = -\frac{g_{44}+1}{2} \tag{4.8}$$

is the known tensor form of the Newtonian potential induced by the mass density ρ as inferred from the linearized Einstein equations, while v is the velocity with respect to the Minkowski metric $\eta_{ab} = g_{ab} - h_{ab}$.

The particle's Lagrangian is thus

$$m_0 \left(\frac{v^2}{2} - \Phi_{\rm N} - \Psi\right), \tag{4.9}$$

which leads to the equation of motion

$$\boldsymbol{a} \approx -\,\partial_\alpha (\Phi_{\rm N} + \Psi). \tag{4.10}$$

Whenever

$$|\partial_\alpha \Psi| \gg |\partial_\alpha \Phi|\,,$$

so (4.10) reduces to (4.1). Thus we obtain the MOND-like dynamics, and also

$$|\partial_\alpha \Psi| \ll a_0\,.$$

In the regime where $|\partial_\alpha \Psi| \gg a_0$, $\mu \approx 1$, $f(y) \approx y$, the quantity Ψ reduces to $\Phi_{\rm N}$.

To keep the particles' acceleration Newtonian, the measurable Newtonian gravitational constant $\mathfrak{G}$ is twice to the bare constant $\mathfrak{G}'$ introduced in (4.4).

However Bekenstein [3] pointed out the setbacks of the relativistic AQUAL: it turns out that the Ψ-waves can propagate faster than light due to the conformal transformation of the physical null cone, and therefore the contribution of Ψ should be kept to a minimum. The last assumption is quite contradicting to the actual galaxies and clusters, which are observed to deflect light stronger than the visible mass.

Disformal related metrics

a) Field $\nabla_a \Psi$. The light deflection problem can be cured by discarding the relation (4.6). It is then suggested to replace the conformal relation by a "disformal" generalized

$$(g_{ab})' = e^{-2\Psi}\left(\mathcal{A}\, g_{ab} + \mathcal{B} L^2\, \nabla_a \Psi\, \nabla_b \Psi\right) \tag{4.11}$$

with $\mathcal{A}$ and $\mathcal{B}$ functions of the invariant $g^{ab} \nabla_a \Psi\, \nabla_b \Psi$, and $L = \frac{1}{a_0}$. This allows to deflect light via the term $\nabla_a \Psi\, \nabla_b \Psi$ of the physical metric.

Here the causality is fully maintained, but it yields smaller light deflection instead of enhancing it.

b) Field $\mathcal{U}^a$. In February 2008, Jacob D. Bekenstein [3] suggested a possible relativistic generalization of the MOND paradigm. This is known as the *Tensor-Vector-Scalar content theory* (in short, Te-Ve-S [3]), which introduces, next to the metric tensor g_{ab}, a timelike four-vector field $\mathcal{U}^a$ and a scalar field ϕ. This vector is normalized so that

$$g^{ab}\,\mathcal{U}_a\mathcal{U}_b = -1\,. \tag{4.12}$$

The physical (real) metric here is obtained by stretching the Einstein metric in the space-time directions orthogonal to $\mathcal{U}^a = g^{ab}\mathcal{U}_b$, by a factor $e^{-2\phi}$, while shrinking it by the same factor in the direction parallel to $\mathcal{U}^a$ according to

$$(g_{ab})' = e^{-2\phi}\left(g_{ab} + \mathcal{U}_a\mathcal{U}_b\right) - e^{2\phi}\mathcal{U}_a\mathcal{U}_b\,. \tag{4.13}$$

When a specific matter content is present, with a density ρ, the *physical* velocity $(u_a)'$ of the matter, normalised with respect to $(g_{ab})'$, is taken to be collinear with $\mathcal{U}_a$

$$(u_a)' = e^{\phi}\,\mathcal{U}_a\,,$$

from which it follows that

$$(g_{ab})' + (u_a)'(u_b)' = e^{-2\phi}\left(g_{ab} + \mathcal{U}_a\mathcal{U}_b\right)\,. \tag{4.14}$$

With these elements, Bekenstein's MOND relativistic theory successfully provides a suitable explanation for mass discrepancy (hypothetical dark matter), and also for several cosmological anomalies without conflicting with GR.

§4.4. Matching the relativistic MOND formulation

Let us consider again the contracted EGR tensors

$$\mathsf{R}_{(ab)} = R_{ab} - \frac{1}{2}\left(g_{ab}\,\nabla_d\,J^d + \frac{1}{3}\,J_aJ_b\right), \tag{4.15}$$

$$\mathsf{R}_{[ab]} = \frac{1}{6}\left(\partial_a\,J_b - \partial_b\,J_a\right), \tag{4.16}$$

where the time components reduce to

$$\mathsf{R}_{(44)} = R_{44} - \frac{1}{2}\left(g_{44}\,\nabla_d\,J^d + \frac{1}{3}\,J_4J_4\right), \tag{4.17}$$

$$\mathsf{R}_{[44]} = 0\,, \tag{4.18}$$

(we note that although, obviously, $\mathsf{R}_{[44]} = 0$ and $J_4 \neq 0$).

For low velocities and weak fields, the quasi-Euclidean approximation holds and $\nabla_d J^d$ is negligible with respect to $\frac{1}{3} J_4 J_4$

$$R_{(44)} = R_{44} - \frac{1}{2}\left(\frac{1}{3} J_4 J_4\right), \tag{4.19}$$

whereas in the classical Newtonian theory

$$R_{44} = \frac{1}{c^2} \frac{\partial^2 \Phi_{\rm N}}{\partial x^\alpha \partial x^\alpha}.$$

Upon the linear approximation, the quantity

$$B_{44} = -\frac{1}{6} J_4 J_4 \tag{4.20}$$

can be identified with the Laplacian of the scalar field Ψ, i.e. with the quantity $-\Delta\Psi$, and we find back the conclusions inferred from the conclusions of the AQUAL model, without recurring to the conformal metric,

$$\boldsymbol{a} \approx -\,\partial_\alpha \left(\Phi_{\rm N} + \Psi\right). \tag{4.21}$$

Causality is therefore respected since no hypothesis is formulated on the light cone structure. As a result, we see that there is no need to introduce the specific (real) metric

$$(g_{ab})' = e^{-2\phi}\left(g_{ab} + \mathcal{U}_a \mathcal{U}_b\right) - e^{2\phi}\, \mathcal{U}_a \mathcal{U}_b\,. \tag{4.22}$$

This purely theoretical approach does not take into account the order of magnitude of the extra curvature which describes the residual field.

Because of this, it may not fit in the relativistic MOND formulation. However we just want to focus our attention onto the fact that the new outlook made possible here by the EGR theory.

Indeed, as we will see in the forthcoming papers, the existence of a persistent field, which is viable through only the EGR theory, provides a sound consistency in other known theories.

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