

# Cosmological Models in the Generalized Einstein Action

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**Abstract:** We have studied the evolution of the Universe in the generalized Einstein action of the form  $R + \beta R^2$ , where  $R$  is the scalar curvature and  $\beta = \text{const}$ . We have found exact cosmological solutions that predict the present cosmic acceleration. These models predict an inflationary de-Sitter era occurring in the early Universe. The cosmological constant ( $\Lambda$ ) is found to decay with the Hubble constant ( $H$ ) as,  $\Lambda \propto H^4$ . In this scenario the cosmological constant varies quadratically with the energy density ( $\rho$ ), i.e.,  $\Lambda \propto \rho^2$ . Such a variation is found to describe a two-component cosmic fluid in the Universe. One of the components accelerated the Universe in the early era, and the other in the present era. The scale factor of the Universe varies as  $a \sim t^n$ ,  $n = 1/2$  in the radiation era. The cosmological constant vanishes when  $n = 4/3$  and  $n = 1/2$ . We have found that the inclusion of the term  $R^2$  mimics a cosmic matter that could substitute the ordinary matter. It is also equivalent to having a bulk viscosity of ordinary cosmology.

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**§1. Introduction.** Modified gravity models have been invoked to resolve cosmological and astrophysical problems with observations (see Hawking & Luttrell [1], Whitt [2], Srivastava [3], Srivastava & Sinha [4], Kung [5]). The generalized Einstein action including an additional scalar term  $R^2$  is given (by Kenmoku *et al.*, 1992 [6]; Nojiri & Odintsov, 2005 [7]; Debnath & Paul, 2006 [8]) by

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{g} (R + 2\Lambda + \beta R^2) + S_{\text{matter}} \quad (1)$$

where  $R$  is Ricci's scalar curvature,  $\Lambda$  is the cosmological constant,  $g$  is the negative determinant of the metric tensor  $g_{\mu\nu}$ ,  $S_{\text{matter}}$  is the

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matter action, and  $\beta$  is a constant. Several authors have studied classical solutions of this action without matter and have concluded that big bang singularity may be avoided (see Kung, 1996 [5]).

In this paper we will study the cosmological implications of this action.

The variation of the metric with respect to  $g_{\mu\nu}$  gives

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} + \beta B_{\mu\nu} = -8\pi G T_{\mu\nu}, \quad (2)$$

where  $T_{\mu\nu}$  is the energy momentum tensor of the cosmic fluid, and

$$B_{\mu\nu} = 2R \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + 2 (R_{;\mu\nu} - g_{\mu\nu} \Delta R), \quad (3)$$

with  $R_{;\mu\nu} = \nabla_\mu \nabla_\nu R$  and  $\Delta R = g^{\mu\nu} R_{;\mu\nu}$ . For an ideal fluid one has

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \quad (4)$$

where  $u_\mu$ ,  $\rho$ ,  $p$  are the velocity, density and pressure of the cosmic fluid.

The flat Robertson-Walker line element is given by

$$ds^2 = -dt^2 + a^2(t) [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (5)$$

The time-time and space-space components of (2) give

$$3H^2 - \Lambda - 18\beta (6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2) = 8\pi G\rho, \quad (6)$$

and

$$-2\dot{H} - 3H^2 + \Lambda + 6\beta (2\ddot{H} + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2) = 8\pi Gp, \quad (7)$$

where  $H = \frac{\dot{a}}{a}$  is the Hubble constant. We have noticed that Debnath and Paul considered a similar action, but with variable  $G$  and  $\Lambda$ . They arrive at very similar type of solutions.

**§2. Model A.** Now consider the cosmological model when

$$\Lambda = -18\beta (6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2) \quad (8)$$

so that (6) is

$$3H^2 = 8\pi G\rho \quad (9)$$

and (7) becomes

$$-2\dot{H} - 3H^2 = 8\pi G \left[ p - \frac{3\beta}{2\pi G} (\ddot{H} + 3H\ddot{H} + 6\dot{H}^2) \right]. \quad (10)$$

A universe with bulk viscosity ( $\eta$ ) is obtained by replacing the pressure  $p$  by the effective pressure  $p - 3\eta H$ . In this case, one may attribute that the inclusion of the  $R^2$  is equivalent to having a bulk viscosity given by

$$\eta = \frac{\beta}{2\pi G} \left( \frac{\ddot{H}}{H} + 3\dot{H} + 6\frac{\dot{H}^2}{H} \right). \quad (11)$$

Notice here that the bulk viscosity is normally parameterized by  $\eta = \eta_0 \rho^s$  ( $s, \eta_0 = \text{const}$ ), but here  $\eta$  depends on the rates of universe expansion. Consider a power law expansion of the form

$$a = A t^n, \quad A, n = \text{const} \quad (12)$$

so that

$$\eta = \frac{3\beta}{\pi G} (2n - 1) t^{-3}. \quad (13)$$

From (8), the cosmological constant becomes

$$\Lambda = 54 \beta n^2 (2n - 1) t^{-4}, \quad n \neq 0, \quad (14)$$

and the energy density

$$8\pi G \rho = \frac{3n^2}{t^2}. \quad (15)$$

Using (12), the cosmological constant becomes

$$\Lambda = \frac{54\beta(2n-1)}{n^2} H^4, \quad n \neq 0, \quad (16)$$

where  $H = \frac{n}{t}$ . Upon using (9), this becomes

$$\Lambda = \frac{6\beta(2n-1)(8\pi G)^2}{n^2} \rho^2, \quad n \neq 0, \quad (17)$$

i.e.,  $\Lambda \propto \rho^2$ . Substituting (12) into (10), we see that the pressure is given by

$$8\pi G p = \left[ \frac{(2-3n)n}{t^2} - \frac{72\beta n(1-2n)}{t^4} \right]. \quad (18)$$

Using (15), this can be written as

$$p = \left( -1 + \frac{2}{3n} \right) \rho - \left( \frac{1-2n}{n^3} \right) N \beta \rho^2, \quad N = 64\pi G, \quad n \neq 0. \quad (19)$$

We know the Van der Waals equation of state is given by

$$p = \frac{\gamma \rho}{1 - b\gamma} - \alpha \rho^2, \quad \gamma, b, \alpha = \text{const}. \quad (20)$$

Thus, the resulting equation of state of a power law expansion is that due to two-component fluid resembling the van der Waals equation of state. Therefore, introducing a term of  $R^2$  in the Einstein action is like introducing two fluid components in the Universe. We see that one component of the fluid drives the Universe in cosmic acceleration, by making  $p < 0$ , in some period and decelerates it in another period ( $p > 0$ ). In the early Universe, when the density was so huge,  $p$  was negative if  $n < \frac{1}{2}$ . During the matter dominated epoch, when the density is very small,  $p < 0$ , if  $n > \frac{2}{3}$ . Hence, we see that the Universe accelerates for any deviation from the Einstein-de-Sitter expansion.

**A. Inflationary Era.** We see from (9) when  $H = H_0 = \text{const}$ , i.e.,  $a \propto \exp(H_0 t)$ , the cosmological constant vanishes, i.e.,  $\Lambda = 0$ . From (11) the bulk viscosity also vanishes, i.e.,  $\eta = 0$ . We recover the de Sitter solution,  $p = -\rho$ , as evident from (9) and (10).

For a static universe  $n = 0$ , i.e.,  $H = 0$ , and hence, the cosmological constant, the bulk viscosity, the energy density and the pressure vanish, i.e.,  $\Lambda = \eta = \rho = p = 0$ . This means that a static universe in this scenario can't exist.

**B. Radiation Dominated Era.** During the radiation dominated phase, as in the Einstein-de-Sitter model, i.e.,  $a \propto t^{\frac{1}{2}}$ , one has  $n = \frac{1}{2}$  so that  $\Lambda = \eta = 0$ , and (19) gives the equation of the state  $p = \frac{1}{3}\rho$ . Thus the Einstein-de-Sitter model is recovered. In this epoch the cosmological constant vanishes. However, (19) shows that any deviation from  $n = \frac{1}{2}$  in the radiation era, viz.,  $n < \frac{1}{2}$ , the second term will be large and negative. Thus, an accelerated expansion of the Universe will be inevitable.

**C. Matter Dominated Era.** In the matter dominated epoch of Einstein-de-Sitter model one has  $n = \frac{2}{3}$ . In this case  $p = 72\pi G\beta\rho^2$ . Since  $\rho$  is small today, we see that the Universe asymptotically approaches the Einstein-de-Sitter type. However, for any deviation of this expansion law,  $n > \frac{2}{3}$  accelerated expansion will be inevitable. In this case  $p < 0$ . So, in the distant future, when  $\rho \rightarrow 0$ , the equation of state reduces to

$$p = \left(-1 + \frac{2}{3n}\right)\rho = \omega\rho, \quad \omega = -1 + \frac{2}{3n}. \quad (21)$$

Thus,  $n > \frac{2}{3}$  implies  $\omega > -1$ . We remark here in the distant future, when  $n \rightarrow \infty$ ,  $p = -\rho$ . Hence, the future of our Universe will be a de-Sitter expansion.

**§3. Model B.** Now, let us define the cosmological constant by

$$\Lambda = -6\beta \left( 2\ddot{H}H + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2 \right), \quad (22)$$

so that (6) and (7) become

$$3H^2 = 8\pi G(\rho + \bar{\rho}), \quad (23)$$

and

$$-2\dot{H} - 3H^2 = 8\pi Gp, \quad (24)$$

where

$$8\pi G\bar{\rho} = -12\beta(\ddot{H} + 3H\ddot{H} + 6\dot{H}^2). \quad (25)$$

**A. Inflationary Era.** We see that when  $H = H_0 = \text{const}$ , i.e.,  $a \propto \exp(H_0 t)$ ,  $p = -\rho$ ,  $\Lambda = 0$  and  $\bar{\rho} = 0$ .

**B. Radiation and Matter Dominated Eras.** Now consider a power law expansion of the scale factor of the form as in (12). We find

$$8\pi Gp = n(2 - 3n)t^{-2}, \quad (26)$$

$$\Lambda = 18\beta n(2n - 1)(3n - 4)t^{-4}, \quad (27)$$

and

$$\rho = \left( \frac{3n}{2-3n} \right) p + N' \left[ \frac{2n-1}{n^2(2-3n)^2} \right] p^2, \quad N' = 637\beta G, \quad n \neq \frac{2}{3}, \quad (28)$$

and

$$8\pi G\bar{\rho} = 72\beta n(1 - 2n)t^{-4}. \quad (29)$$

Equation (28) represents our equation of state for the present cosmology. The cosmological constant here varies as  $t^{-4}$ .

The equation of state now reads,

$$p = \omega(t)\rho, \quad \omega(t) = \left[ \frac{3n}{2-3n} + 72\beta \frac{(2n-1)}{n(2-3n)} \frac{1}{t^2} \right]^{-1}, \quad (30)$$

where the energy density becomes

$$8\pi G\rho = \frac{3n^2}{t^2} \left[ 1 + \frac{24\beta(2n-1)}{n} \frac{1}{t^2} \right]. \quad (31)$$

It is evident from (30) that when  $n \rightarrow \infty$  (i.e.,  $a \rightarrow \infty$ ),  $\omega \rightarrow -1$  the Universe becomes vacuum dominated and expands like de-Sitter. It is

interesting to note that when  $n = \frac{4}{3}$ , the cosmological constant vanishes, i.e.,  $\Lambda = 0$ . In this case the pressure becomes negative, i.e.,  $p < 0$ , and this drives the Universe into an epoch of cosmic acceleration. The deceleration parameter  $q = -\frac{\ddot{a}}{aH^2} = -0.25$ . Once again, when  $n = \frac{1}{2}$ , we recover the Einstein-de-Sitter solution, i.e.,  $a \propto t^{\frac{1}{2}}$  and  $p = \frac{1}{3}\rho$  and  $\Lambda = 0$ . For  $n = \frac{2}{3}$ ,  $p = 0$ ,  $\Lambda = -8\beta t^{-4}$ , and  $8\pi G\rho = \frac{4}{3t^2}\left(1 + \frac{12\beta}{t^2}\right)$ . Hence, the Universe approaches the Einstein-de-Sitter solution asymptotically ( $t \rightarrow \infty$ ). Notice that (27) and (29) relate the vacuum energy density  $\rho_v = \frac{\Lambda}{8\pi G}$  to  $\bar{\rho}$  by the equation  $\rho_v = \left(1 - \frac{3}{4}n\right)\bar{\rho}$ , so that for  $n = \frac{2}{3}$ ,  $\bar{\rho} = 2\rho_v$ .

**§4. Model C.** Now consider a cosmological model in which  $\Lambda = 0$ . In this case, (6) and (7) yield

$$3H^2 = 8\pi G(\rho + \rho') \quad (32)$$

and

$$-2\dot{H} - 3H^2 = 8\pi G(p + p'), \quad (33)$$

where

$$\left. \begin{aligned} 8\pi G\rho' &= 18\beta(6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2) \\ 8\pi Gp' &= -6\beta(2\ddot{H} + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2) \end{aligned} \right\}, \quad (34)$$

equations (32) and (33) can be written as

$$3H^2 = 8\pi G\rho_{\text{eff}} \quad (35)$$

and

$$-2\dot{H} - 3H^2 = 8\pi Gp_{\text{eff}}, \quad (36)$$

where

$$\rho_{\text{eff}} = \rho + \rho', \quad p_{\text{eff}} = p + p'. \quad (37)$$

We therefore argue that the inclusion of the term  $R^2$  in the Einstein action induces a fluid in the Universe that has pressure ( $p'$ ) and energy density ( $\rho'$ ), in addition to the pre-existing matter. This may suggest that our Universe is filled with a fluid with two components; one is *bright* ( $\rho$ ) and the other is *dark* ( $\rho'$ ) without having a cosmological constant.

**A. Inflationary Era.** An inflationary solution arises when  $H = H_0 = \text{const}$  which is solved to give  $a \propto \exp(H_0 t)$ . Equations (35) and (36) give

$$p_{\text{eff}} = -\rho_{\text{eff}}. \quad (38)$$

With some scrutiny, one would discover that (34) implies that  $p' = \rho' = 0$ . Hence, the *dark* component does not contribute to this inflationary era.

**B. Radiation Dominated Era.** Now consider a power law expansion for the Universe during the radiation dominated era of the form

$$a = Bt^n, \quad n, B = \text{const}. \quad (39)$$

Substituting this into (34) one gets

$$\rho' = \frac{54\beta n^2(1-2n)}{8\pi G} t^{-4}, \quad (40)$$

and

$$p' = \frac{18\beta n(1-2n)(4-3n)}{8\pi G} t^{-4}. \quad (41)$$

These two equations are related by the relation

$$p' = \left(-1 + \frac{4}{3n}\right) \rho', \quad n \neq 0, \quad (42)$$

which represents the equation of state of the *dark* fluid. It is very interesting to note that, when  $n = \frac{1}{2}$ ,  $\rho' = p' = 0$ . This implies that the *dark* component does not disturb the nucleosynthesis constraints set forth by the Einstein-de-Sitter solution. Notice that when  $n \rightarrow \infty$ ,  $p' = -\rho'$ , so that in the distant future the Universe with or without normal matter will be vacuum dominated. We notice that this *dark* component does not live in a static Universe since it has  $p' = \rho' = 0$ . For a positive energy density, we have (for  $\beta > 0$ ) the constraint  $n < \frac{1}{2}$ . For  $\frac{1}{2} < n < \frac{4}{3}$ ,  $\rho' < 0$ ,  $p' < 0$ . When  $n = 1$ ,  $p' = \frac{1}{3}\rho'$  so that the dark component mimics the ordinary radiation. The positivity of energy density is recovered if  $\beta < 0$ . It is remarkable to notice that if the matter action is not incorporated in the Einstein action, the inclusion of the quadratic term acts like matter. This type of matter is characterized by its equation of state in (42). Hence, the inclusion of the  $R^2$  mimics the introduction of a new matter in the Universe.

**C. Matter Dominated Era.** Applying (39)–(41) in (32) and (33), one gets

$$8\pi G\rho = \frac{3n^2}{t^2} \left[1 - \frac{18\beta(1-2n)}{t^2}\right], \quad (43)$$

and

$$8\pi Gp = \frac{n(2-3n)}{t^2} \left[1 - \frac{18\beta(2n-1)(3n-4)}{(2-3n)} \frac{1}{t^2}\right]. \quad (44)$$

Once again, we see from (41) that the pressure of the *dark* fluid vanishes when  $n = \frac{4}{3}$  leaving only the *bright* fluid to contribute to the

universal pressure. In this case,  $p < 0$ , and this will drive the Universe into a cosmic acceleration era. Substitution of  $n = \frac{2}{3}$  in (43) and (44) yields

$$8\pi G\rho = \frac{4}{3t^2} \left( 1 + \frac{6\beta}{t^2} \right), \quad 8\pi Gp = \frac{8\beta}{t^4}. \quad (45)$$

We remark that the Universe approaches the Einstein-de-Sitter asymptotically (when  $t \rightarrow \infty$ , i.e., in the distant future), where  $\rho = \frac{1}{6\pi Gt^2}$  and  $p = 0$ . However,  $\rho_{\text{eff}} = \frac{1}{6\pi Gt^2}$  and  $p_{\text{eff}} = 0$ . Notice that during this era the *dark* component behaves like a stiff matter, with  $p' = \rho'$ . It is evident that when  $\frac{2}{3} < n < \frac{4}{3}$  the Universe enters a period of cosmic acceleration. We may thus argue that the present observed cosmic acceleration happened during this period. Equation (45) looks like having a cosmological constant of the form  $\Lambda \propto H^4$  in the Einstein-de-Sitter universe.

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