The Classification of Spaces Defining Gravitational Fields

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Abstract: In this paper written in 1954 Alexei Petrov describes his famous classification of spaces according to the algebraical structure of the curvature tensor, that determines the classes of the gravitational fields permitted therein. Now this classification of spaces (and, respectively, of the gravitational fields) is known as *Petrov's classification*. This paper was originally published, in Russian, in Scientific Transactions of Kazan State University: Petrov A. Z. Klassifikazija prostranstv, opredelajuschikh polja tjagotenia. *Uchenye Zapiski Kazanskogo Gosudarstvennogo Universiteta*, 1954, vol. 114, book 8, pages 55–69. Translated from Russian in 2008 by Vladimir Yershov, England–Pulkovo.

In this paper, the detailed proof of results obtained and published by the author earlier in 1951 [1]. Namely, it is shown that by examining the algebraic structure of the curvature tensor V_4 one can establish a classification of the gravitational fields defined by this tensor and given in the form

$$ds^2 = g_{ij}\, dx^i dx^j, \tag{1}$$

with the fundamental tensor satisfying the field equations

$$R_{ij} = \varkappa\, g_{ij} \tag{2}$$

(we shall refer to the corresponding manifolds as T_4).

§1. Bivector space. Let us consider a point P of the manifold T_4, and associate it with a local center-affine geometry E_4. In this E_4 let us select those tensors that satisfy the following conditions: 1) the number of both covariant and contravariant indices must be even; and 2) the covariant and contravariant indices can be grouped in separate antisymmetric pairs. We shall regard each of these pairs as a single collective index, denoting it with a Greek letter in order to distinguish it from the indices corresponding to T_4 and E_4, for which we shall continue using Latin letters. Thus, according to the number of possible values for these collective indices, we shall get an $N = \frac{n(n-1)}{2}$ - dimensional manifold (6 dimensions for $n = 4$), the tensors E_4 with these properties defining on this manifold tensors with one-half rank.

One can say that *each point of T_4 is assigned to a local 6-dimensional centre-affine geometry with the group*

$$\left.\begin{array}{ll} \eta^{\alpha'} = A^{\alpha'}_{\alpha}\,\eta^{\alpha}, & \eta^{\alpha} = A^{\alpha}_{\alpha'}\,\eta^{\alpha'} \\ |A^{\alpha'}_{\alpha}| \neq 0\,, & A^{\alpha}_{\beta}A^{\beta}_{\gamma} = \delta^{\alpha}_{\gamma} \end{array}\right\}. \tag{3}$$

Indeed, by ordering the collective indices (while selecting a single pair from the two possible, ij and ji), we shall get six possible collective indices. Let us take, for example, the following indexing:

$$1 - 14,\quad 2 - 24,\quad 3 - 34,\quad 4 - 23,\quad 5 - 31,\quad 6 - 12.$$

Let us now consider the transformation of the components T^{ij} of, generally speaking, a nonsimple bivector

$$T^{i'j'} = A^{i'j'}_{ij}\,T^{ij},$$

assuming

$$A^{\alpha'}_{\alpha} = 2A^{[i'j']}_{ij}, \quad \text{where } A^{i'}_{i} = \left(\frac{\partial x^{i'}}{\partial x^{i}}\right)_{P}.$$

In terms of collective indices, this gives

$$T^{\alpha'} = A^{\alpha'}_{\alpha}\,T^{\alpha};$$

i.e., the set of bivectors T_n determines a set of contravariant vectors in E_N (in this case the dimensionality does not matter), assuming that the relations (3) are satisfied. The validity of these relations can be checked directly by passing to the Latin indices.

Let us call the manifold obtained a *bivector space*. Of a special interest for our further consideration will be the curvature tensor T_4. In the bivector space this tensor corresponds to a symmetric tensor of the second rank

$$R_{ijkl} \longrightarrow R_{\alpha\beta} = R_{\beta\alpha}\,.$$

In any local E_6 one can define a metric by using for this purpose any tensor in T_4 with the properties

$$M_{klij} = M_{jikl} = -M_{ijkl} = -M_{ijlk}\,,$$

given that the corresponding second-rank tensor in E_6 is nonsingular. Let the tensor

$$g_{ikjl} = g_{ij}g_{kl} - g_{il}g_{kj} \longrightarrow g_{\alpha\beta} = g_{\beta\alpha} \tag{4}$$

be such a fundamental tensor in E_6. It is plain to see that $g_{\alpha\beta}$ gives a nondegenerate metrization because $|g_{ij}| \neq 0$, and

$$|g_{\alpha\beta}| = p\,|g_{ij}|^{2n}, \qquad p \neq 0\,.$$

For a definite g_{ij} the tensor $g_{\alpha\beta}$ will be definite; and for an indefinite g_{ij} the tensor $g_{\alpha\beta}$ will also, in general, be indefinite. Let us note, that here we shall consider only those fields of gravity that correspond to a real distribution of matter in space, which would require [2] the fundamental tensor g_{ij} be reducible to the form

$$(g_{ij}) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \tag{5}$$

in the real coordinate system in any given point of T_4, that is, we have arrived at the so-called Minkowski space. Then it follows from (4) that for the frame corresponding to the matrix (5) the fundamental tensor R_6 will be of the following form:

$$(g_{\alpha\beta}) = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \qquad |g_{\alpha\beta}| = -1, \tag{6}$$

i.e., the tensor $g_{\alpha\beta}$ is, in fact, indefinite.

§2. Classification of T_4. A series of the most interesting problems arising in the study of the Riemannian manifolds is related to the curvature tensor V_n. As is known, this tensor is used for introducing the notion of curvature of V_n at a given point along a given two-dimensional direction or, which is the same, of the Gaussian curvature of a two-dimensional geodesic surface at a given point:

$$K = \frac{R_{ijkl}V^{ij}V^{kl}}{g_{pqrs}V^{pq}V^{rs}}\,; \tag{7}$$

where g_{pqrs} has the form (4), and the two-dimensional direction, which is defined by the vectors $\underset{1}{V}^i$ and $\underset{2}{V}^i$, is characterized by the simple bivector $V^{ij} = \underset{[1}{V}^i \underset{2]}{V}^j$. Let us introduce the notion of *generalized curvature* of V_n, which could be obtained from (7) by dropping the requirement of simplicity of the bivector V^{ij}. At some point of V_n this generalized invariant K will be a homogeneous zero-degree function of the components of the (generally, not simple) bivector V^{ij}. And, of course, this invariant will be meaningful in the bivector space, where it can be written as

$$K = \frac{R_{\alpha\beta}\,V^{\alpha}V^{\beta}}{g_{\alpha\beta}\,V^{\alpha}V^{\beta}}\,. \tag{8}$$

Let us find the critical values of K that will be equivalent to finding those vectors V^{α} in R_N, for which K takes critical values. Let us call these critical values of K *stationary curvatures* of V_n, and the corresponding bivectors V^{α} — *the stationary directions* in V_n. Thus, our task consists in finding *the unconditionally stationary vectors* V^{α} in the bivector space using the necessary and sufficient conditions for stationarity:

$$\frac{\partial K}{\partial V^{\alpha}} = 0\,. \tag{9}$$

We have to take into account that for an indefinite g_{ij} the tensor $g_{\alpha\beta}$ is also indefinite and, hence, it is possible to have isotropic stationary directions

$$g_{\alpha\beta}\,V^{\alpha}V^{\beta} = 0\,. \tag{10}$$

Let us first exclude this case, returning to it below.

If (10) does not hold then the conditions (9) result in

$$(R_{\alpha\beta} - K g_{\alpha\beta})\,V^{\beta} = 0\,, \tag{11}$$

i.e., the stationary directions of V_n will be the principal axes of the tensor $R_{\alpha\beta}$ in the bivector space, while the stationary curvatures of V_n will be the characteristic values of the secular equation

$$|\,R_{\alpha\beta} - K g_{\alpha\beta}\,| = 0\,. \tag{12}$$

Let (10) holds now for the stationary V^{α}. Since we are interested only in the K satisfying the conditions (9), this K is a continuous function of V^{α} and, hence, it is necessary that the condition

$$R_{\alpha\beta}\,V^{\alpha}V^{\beta} = 0$$

were satisfied. Then one can calculate the value of K for the stationary isotropic direction of V^{α}:

$$K\,(V^{\alpha}) = \lim_{dV^{\alpha}\to 0} K\,(V^{\alpha} + dV^{\alpha})\,,$$

assuming the continuity of K as a function of V^α. If, for a given V^α, we denote

$$\varphi = g_{\alpha\beta}\, V^\alpha V^\beta, \qquad \psi = R_{\alpha\beta}\, V^\alpha V^\beta, \tag{13}$$

then for a stationary isotropic V^α

$$K(V^\alpha) = \lim_{dV^\alpha \to 0} \frac{\psi(V^\alpha + dV^\alpha) - \psi(V^\alpha)}{\varphi(V^\alpha + dV^\alpha) - \varphi(V^\alpha)} = \lim \frac{\Sigma_\sigma \frac{\partial}{\partial V^\sigma}\,\psi\, dV^\sigma + \dots}{\Sigma_\sigma \frac{\partial}{\partial V^\sigma}\,\varphi\, dV^\sigma + \dots}\,.$$

As this limit cannot depend on the ways of changing dV^α, then

$$K(V^\alpha) = \frac{\frac{\partial}{\partial V^\sigma}\,\psi}{\frac{\partial}{\partial V^\sigma}\,\varphi} = \frac{R_{\sigma\beta}\, V^\beta}{g_{\sigma\beta}\, V^\beta}\,,$$

so that again we obtain (11).

The determination of stationary curvatures and directions in R_N leads to the study of the pair of the quadratic forms (13). Therefore, the reduction of this pair to canonical form in real space results in a classification for the curvature tensor of V_n at a given point of V_n, as well as in a neighboring plane containing this point, where *the characteristic of the K-matrix*

$$\|\, R_{\alpha\beta} - K g_{\alpha\beta} \,\| \tag{14}$$

remains constant. For each type of the characteristic (14) there is a corresponding field of gravity of a specific type. It is this that determines the sought classification of T_4.

Using real transformations, one can always reduce the matrix $\|\, g_{\alpha\beta}\|$ to the form (6), and it remains to simplify the matrix $\|\, R_{\alpha\beta}\|$ by using real orthogonal transformations.

Theorem 1. *The matrix $\|\, R_{\alpha\beta}\|$ will be symmetrically-double for the orthogonal frame (5).*

For the basic (5) the field equations will take the form

$$\sum_k e_k\, R_{ikjk} = \varkappa\, g_{ij}\,, \qquad e_k = \pm 1\,,$$

that is, for $i = j$

$$\sum_k e_k\, R_{ikik} = \varkappa\, e_i\,,$$

and for $i \neq j$

$$e_k\, R_{ikjk} + e_l\, R_{iljl} = 0 \qquad (i, j, k, l \neq)\,.$$

Writing these relations with the use of collective indices of the bivector space and taking into account the indexing introduced in § 1, we shall get the following expression for our matrix:

$$\left.\begin{array}{c} \| R_{\alpha\beta} \| = \left\| \begin{array}{c|c} M & N \\ \hline N & -M \end{array} \right\| \\ M = \left\| \begin{array}{ccc} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{array} \right\|, \quad m_{\alpha\beta} = m_{\beta\alpha} \\ N = \left\| \begin{array}{ccc} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{array} \right\|, \quad n_{\alpha\beta} = n_{\beta\alpha} \\ (\alpha, \beta = 1, 2, 3) \end{array} \right\}, \tag{15}$$

where $\sum_{i=1}^{3} m_{ii} = \varkappa$ and $\sum_{i=1}^{3} n_{ii} = 0$, due to the Ricci identity, which proves the theorem. Let us note that similar matrices were obtained by V. F. Kagan [3], when studying the group of Lorentz transformations, although he used a condition of orthogonality of these matrices. Under the same assumption of orthogonality, similar matrices were also studied by Ya. S. Dubnov [4] and A. M. Lopshitz [5]. The fact established by the previous theorem takes place for any orthogonal frame and, hence, taking into account that the orthogonal frame has 6 degrees of freedom for $n = 4$, one can expect the possibility of further simplification of the matrix by choosing 6 appropriate rotations.

First let us prove a theorem that would essentially narrow down the number of possible (at first sight) types of the characteristic of the matrix (14).

Theorem 2. *The characteristic of the matrix (14) always consists of two identical parts.*

Let us reduce the matrix (14) to a simpler form by using the so-called elementary transformations, which, as is known, do not change the elementary divisors of a matrix and, therefore, its characteristic. Let us represent this matrix in the following way:

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + K\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline n_{\alpha\beta} & -\, m_{\alpha\beta} - K\delta_{\alpha\beta} \end{array} \right\|,$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. By multiplying the last column by i and adding it to the corresponding first column we shall get the equivalent matrix

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline -\, i\,(m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta}) & -\, m_{\alpha\beta} - K\delta_{\alpha\beta} \end{array} \right\|.$$

By multiplying the first row of the previous matrix by i and adding it to the last row we shall convert the matrix to the form

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta} & n_{\alpha\beta} \\ \hline 0 & -m_{\alpha\beta} + i n_{\alpha\beta} - K\delta_{\alpha\beta} \end{array} \right\|.$$

Finally, by multiplying the first column by $\frac{i}{2}$ and adding it to the corresponding last column and making the same operation with the last row, we shall obtain the matrix

$$\left\| \begin{array}{c|c} m_{\alpha\beta} + i n_{\alpha\beta} + K\delta_{\alpha\beta} & 0 \\ \hline 0 & m_{\alpha\beta} - i n_{\alpha\beta} + K\delta_{\alpha\beta} \end{array} \right\| \equiv \left\| \begin{array}{c|c} P(K) & 0 \\ \hline 0 & \overline{P}(K) \end{array} \right\|, \quad (16)$$

which is equivalent to the K-matrix (14). The task has been reduced to the studying of two three-dimensional matrices $P(K)$ and $\overline{P}(K)$, whose corresponding elements are complex-conjugate. It follows then that the elementary divisors of these two matrices are also complex-conjugate and, hence, their characteristics have the same form. Therefore, the characteristic of our K-matrix consists of two parts repeating each other, so that the theorem holds.

Let us note that the principal directions and invariant bundles of the K-matrix should also be pairwisely complex-conjugate.

Now we can accomplish the classification of the fields of gravity. This classification can be expressed through the following theorem.

Theorem 3. *There exist three and only three types of the fields of gravity.*

The three-dimensional matrix $P(K)$ can have only one of three possible types of characteristic: $[1\ 1\ 1]$, $[2\ 1]$, $[3]$, if we neglect the cases when some of the elementary divisors have the same basis and, thus, some of the numbers in the square brackets should be enclosed in parentheses, e.g., $[(1\,1)\,1]$, $[(2\,1)]$, etc.

The characteristic of $\overline{P}(K)$ will have the same form. Then the characteristics of the K-matrix will be written as following:

$$1)\ [1\overline{1},\ 1\overline{1},\ 1\overline{1}]; \quad 2)\ [2\overline{2},\ 1\overline{1}]; \quad 3)\ [3\ \overline{3}],$$

where the overlined numbers correspond to the power index of the elementary divisor with the basis being complex-conjugate to the basis whose power index is expressed by the previous number.

Each of these types of the gravity fields has to be considered separately; and of a prime importance here is to get the canonical forms of the matrix $\| R_{\alpha\beta} \|$ for each of these types.

§3. The canonical form of the matrix $\| R_{\alpha\beta} \|$. Let us consider the first type with the characteristic $[1\,\overline{1},\, 1\overline{1},\, 1\overline{1}\,]$. As in this case the characteristic is of simple type, the tensor $R_{\alpha\beta}$ has 6 non-isotropic, pairwisely orthogonal principal directions [6]. One can show that at a given point of T_4 these directions of the bivector space will give the bivectors of specific structure.

Let us denote the vector components of the real orthogonal frame at a point of T_4 by $\underset{k}{\xi}{}^{i}$ $(k, i = 1, \ldots, 4)$, denoting for brevity by $\underset{kl}{\xi}{}^{ij}$ the simple bivectors $\underset{[k}{\xi}{}^{i}\underset{l]}{\xi}{}^{j}$ $(k \neq l)$ that determine the two-dimensional plane corresponding to the vectors of the frame. In the bivector space, these simple bivectors define 6 non-isotropic, mutually independent and orthogonal coordinate vectors $\underset{\sigma}{\xi}{}^{\alpha} = \delta^{\alpha}_{\sigma}$, so that any vector in R_6 (in particular, the vectors of the principal directions in $R_{\alpha\beta}$) can be represented in terms of these vectors.

Let us show that we can take the vectors

$$W^{\alpha} = \lambda\,(\underset{1}{\xi}{}^{\alpha} \pm i\,\underset{4}{\xi}{}^{\alpha}) + \mu\,(\underset{2}{\xi}{}^{\alpha} \pm i\,\underset{5}{\xi}{}^{\alpha}) + \nu\,(\underset{3}{\xi}{}^{\alpha} \pm i\,\underset{6}{\xi}{}^{\alpha}) \tag{17}$$

as the vectors of principal directions, which are uniquely defined only in the case when the roots of the secular equation (12) are all distinct.

Indeed, the condition of W^{α} to define the principal direction of the tensor $R_{\alpha\beta}$ is written as

$$(R_{\alpha\beta} - K g_{\alpha\beta})\, W^{\beta} = 0\,. \tag{18}$$

But due to the symmetric twoness of the K-matrix this system of six equations can be reduced to three equations

$$(m_{s1} \pm i\,n_{s1} + k)\,\lambda + (m_{s2} \pm i\,n_{s2})\,\mu + (m_{s3} \pm i\,n_{s3})\,\nu = 0\,, \quad s = 1, 2, 3\,.$$

For λ, μ, ν to be the non-zero solutions of this system it is necessary and sufficient that K were the root of one of the equations

$$|\,P(K)| = 0\,, \qquad |\,\overline{P}(K)| = 0\,, \tag{19}$$

i.e., a root of the secular equation (12), which proves the theorem.

At a given point of T_4 the vector W^{α} (17) of the manifold R_6 corresponds to the bivector of completed rank:

$$W^{ij} = \lambda\,(\underset{14}{\xi}{}^{ij} \pm i\,\underset{23}{\xi}{}^{ij}) + \mu\,(\underset{24}{\xi}{}^{ij} \pm i\,\underset{31}{\xi}{}^{ij}) + \nu\,(\underset{34}{\xi}{}^{ij} \pm i\,\underset{12}{\xi}{}^{ij})\,. \tag{20}$$

One can easily check that, under any (real) orthogonal transformation, W^{ij} grades into a bivector of the same type, with $\lambda, \mu, \nu \longrightarrow \overset{*}{\lambda}, \overset{*}{\mu}, \overset{*}{\nu}$, so that the norm of the bivector remains invariant:

$$\lambda^2 + \mu^2 + \nu^2 = \overset{*}{\lambda}{}^2 + \overset{*}{\mu}{}^2 + \overset{*}{\nu}{}^2.$$

Let the roots of (12) K $(s = 1, 2, 3)$ correspond to the vectors of the principal direction $\underset{s}{W}{}^\alpha$; then, according to the above reasoning, the roots $\underset{s+3}{K}$ should correspond to $\overline{\underset{s}{W}}{}^\alpha$, provided the appropriate indexing of the roots.

The root $\underset{1}{K}$ corresponds to the bivector

$$\underset{1}{W}{}^{pq} = \underset{1}{\lambda}(\underset{14}{\xi}{}^{pq} + i\,\underset{23}{\xi}{}^{pq}) + \underset{1}{\mu}(\underset{24}{\xi}{}^{pq} + i\,\underset{31}{\xi}{}^{pq}) + \underset{1}{\nu}(\underset{34}{\xi}{}^{pq} + i\,\underset{12}{\xi}{}^{pq}),$$

and the root $\underset{4}{K}$ corresponds to the bivector

$$\underset{4}{W}{}^{pq} = \underset{1}{\overline{\lambda}}(\underset{14}{\xi}{}^{pq} - i\,\underset{23}{\xi}{}^{pq}) + \underset{1}{\overline{\mu}}(\underset{24}{\xi}{}^{pq} - i\,\underset{31}{\xi}{}^{pq}) + \underset{1}{\overline{\nu}}(\underset{34}{\xi}{}^{pq} - i\,\underset{12}{\xi}{}^{pq}).$$

Let us represent the bivector W^{pq} as a sum of two real bivectors $\underset{1}{V}{}^{pq} + i\,\underset{1}{\overset{*}{V}}{}^{pq}$. Then

$$\underset{4}{W}{}^{pq} = \underset{1}{V}{}^{pq} - i\,\underset{1}{\overset{*}{V}}{}^{pq}.$$

Let

$$\lambda = \underset{1}{a} + i\,\underset{1}{b}, \qquad \mu = \underset{2}{a} + i\,\underset{2}{b}, \qquad \nu = \underset{3}{a} + i\,\underset{3}{b},$$

where $\underset{s}{a}$, $\underset{s}{b}$ are real numbers $(s = 1, 2, 3)$; hence

$$\underset{1}{V}{}^{pq} = \underset{1}{a}\,\underset{14}{\xi}{}^{pq} + \underset{2}{a}\,\underset{24}{\xi}{}^{pq} + \underset{3}{a}\,\underset{34}{\xi}{}^{pq} - \underset{1}{b}\,\underset{23}{\xi}{}^{pq} - \underset{2}{b}\,\underset{31}{\xi}{}^{pq} - \underset{3}{b}\,\underset{12}{\xi}{}^{pq},$$

$$\underset{1}{\overset{*}{V}}{}^{pq} = \underset{1}{b}\,\underset{14}{\xi}{}^{pq} + \underset{2}{b}\,\underset{24}{\xi}{}^{pq} + \underset{3}{b}\,\underset{34}{\xi}{}^{pq} - \underset{1}{a}\,\underset{23}{\xi}{}^{pq} - \underset{2}{a}\,\underset{31}{\xi}{}^{pq} - \underset{3}{a}\,\underset{12}{\xi}{}^{pq}.$$

Since $\underset{1}{W}{}^\alpha$ is not an isotropic vector of R_6, then it can always be regarded as a unit vector

$$g_{\alpha\beta}\,\underset{1}{W}{}^\alpha \underset{1}{W}{}^\beta = 1,$$

which leads us to the conclusion that

$$\sum_{s=1}^{3} \underset{s}{a}\underset{s}{b} = 0, \tag{21}$$

$$\sum_{s=1}^{3} \underset{s}{b}{}^2 - \underset{s}{a}{}^2 > 0. \tag{22}$$

Now we can assert the following.

1. The real bivectors $\underset{1}{V}^{pq}$ and $\underset{1}{\overset{*}{V}}{}^{pq}$ are single-foliated. Indeed, by writing down the simplicity condition we shall arrive at (21).
2. They are *0-parallel*. They cannot be be $\frac{2}{2}$-parallel, which would be possible only when the coefficients were proportional at equal $\underset{ij}{\xi}{}^{pq}$; then they would have to be equal to zero. For example,

$$\frac{\underset{1}{a}}{\underset{1}{b}} = -\frac{\underset{1}{b}}{\underset{1}{a}}\,, \qquad \underset{1}{a}^2 + \underset{1}{b}^2 = 0\,.$$

   They cannot be $\frac{1}{2}$-parallel either, as in this case $\underset{1}{W}^{\alpha}$ would be a single-foliated complex bivector; but then by writing the simplicity condition we would arrive at a contradiction with (21) and (22). Therefore, we are left only with the above possibility.
3. These bivectors are $\frac{2}{2}$-perpendicular. For this to be true, it is necessary and sufficient to satisfy the equalities

$$\underset{1}{V}_{is}\underset{1}{\overset{*}{V}}{}^{sj} = 0$$

   for any i, j. It is plain to see that these equalities are reduced to (21), so that they are, indeed, satisfied.

Let us consider a simple bivector $\underset{1}{V}^{pq}$. Its norm, according to (22), is

$$g_{\alpha\beta}\underset{1}{V}^{\alpha}\underset{1}{V}^{\beta} = \sum \underset{s}{b}^2 - \underset{s}{a}^2 > 0\,.$$

In the plain of this real bivector, one can always chose two real, orthogonal and non-isotropic vectors η^p, ν^p. Then the norm of our bivector can also be expressed in the form

$$2\,\eta_p\eta^p\,\nu_q\nu^q\,,$$

and, hence, these two vectors are both either *space-like* or *time-like*. Their norms cannot be > 0, because if we took these two real orthogonal vectors as coordinate vectors, we would arrive at a contradiction with the law of inertia of quadratic forms. Therefore, these two vectors have negative norms. Due to this, by re-normalizing them, we can take them as the vectors $\underset{2}{\overset{*}{\xi}}{}^i$, $\underset{3}{\overset{*}{\xi}}{}^i$ of a new real orthogonal frame.

In a similar way, let us define in the plane $\underset{1}{\overset{*}{V}}{}^{pq}$ two orthogonal (mutually and with respect to $\underset{2}{\overset{*}{\xi}}{}^i$, $\underset{3}{\overset{*}{\xi}}{}^i$) vectors, which will be real and non-isotropic but already having the norms of opposite signs, since

$$g_{\alpha\beta}\,\underset{1}{\overset{*}{\nu}}{}^{\alpha}\underset{1}{\overset{*}{\nu}}{}^{\beta} < 0\,.$$

Let us denote these vectors as $\overset{*}{\underset{1}{\xi}}{}^i$ and $\overset{*}{\underset{4}{\xi}}{}^i$. In this coordinate system

$$\overset{*}{\underset{1}{W}}{}^{pq} = \underset{14}{\xi}{}^{pq} + i\,\underset{23}{\xi}{}^{pq}\,,$$
$$\overset{*}{\underset{4}{W}}{}^{pq} = \underset{14}{\xi}{}^{pq} - i\,\underset{23}{\xi}{}^{pq}\,.$$

Let us note that the frame $\{\overset{*}{\xi}\}$ has been chosen up to a rotation in the plane $\{\overset{*}{\underset{2}{\xi}}\overset{*}{\underset{3}{\xi}}\}$ and a Lorentz rotation in the plane $\{\overset{*}{\underset{1}{\xi}}\overset{*}{\underset{4}{\xi}}\}$. Of course, we are interested in the bivectors $\underset{\sigma}{W}{}^{pq}$ only up to a scalar factor.

Now, writing the orthogonality condition for $\overset{*}{\underset{1}{W}}{}^{pq}$ and $\overset{*}{\underset{2}{W}}{}^{pq}$, we find, of course, that the bivector of the second principal direction should have the form

$$\overset{*}{\underset{2}{W}}{}^{pq} = \overset{*}{\underset{2}{\mu}}(\overset{*}{\underset{24}{\xi}}{}^{pq} + i\,\overset{*}{\underset{31}{\xi}}{}^{pq}) + \overset{*}{\underset{2}{\nu}}(\overset{*}{\underset{34}{\xi}}{}^{pq} + i\,\overset{*}{\underset{12}{\xi}}{}^{pq})\,.$$

Let us make use of the above indicated arbitrariness in the choice of the frame and perform the following rotations:

$$\underset{1}{\xi}{}^p = \operatorname{ch}\varphi\,\overset{*}{\underset{1}{\xi}}{}^p + \operatorname{sh}\varphi\,\overset{*}{\underset{4}{\xi}}{}^p\,,$$
$$\underset{4}{\xi}{}^p = \operatorname{sh}\varphi\,\overset{*}{\underset{1}{\xi}}{}^p + \operatorname{ch}\varphi\,\overset{*}{\underset{4}{\xi}}{}^p\,,$$
$$\underset{2}{\xi}{}^p = \cos\psi\,\overset{*}{\underset{2}{\xi}}{}^p + \sin\psi\,\overset{*}{\underset{3}{\xi}}{}^p\,,$$
$$\underset{3}{\xi}{}^p = -\sin\psi\,\overset{*}{\underset{2}{\xi}}{}^p + \cos\psi\,\overset{*}{\underset{3}{\xi}}{}^p\,.$$

After these transformations $\underset{1}{W}$ will have the same form; hence $\underset{2}{W}$ will also be expressed as

$$\underset{2}{\widetilde{W}}{}^{pq} = \underset{2}{\tilde{\mu}}(\underset{24}{\tilde{\xi}}{}^{pq} + i\,\underset{31}{\tilde{\xi}}{}^{pq}) + \underset{2}{\tilde{\nu}}(\underset{34}{\tilde{\xi}}{}^{pq} + i\,\underset{12}{\tilde{\xi}}{}^{pq})\,,$$

where

$$\underset{2}{\tilde{\nu}} = \sin\psi\operatorname{ch}\varphi + p\cos\psi\operatorname{ch}\varphi + q\sin\psi\operatorname{sh}\varphi + {} + i\,(\cos\psi\operatorname{sh}\varphi + q\cos\psi\operatorname{ch}\varphi - p\sin\psi\operatorname{sh}\varphi)\,,$$

$$p + i\,q = \frac{\overset{*}{\underset{2}{\nu}}}{\overset{*}{\underset{2}{\mu}}}\,,$$

and $\overset{*}{\underset{2}{\mu}}$ can be considered not being equal to zero, otherwise we would be satisfied with the values $\varphi = \psi = 0$. One can find real φ and ψ for any

$\underset{2}{\tilde{\nu}} = 0$. Now the frame is defined uniquely, and, if the orthogonality of $\underset{1}{W}$, $\underset{2}{W}$, $\underset{3}{W}$ is taken into account, the bivectors will have the following form in this frame (up to a scalar factor):

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq}\,,$$
$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}\,,$$
$$\underset{3}{W}^{pq} = \underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}\,,$$

and, due to the mentioned above complex conjugacy,

$$\underset{4}{W}^{pq} = \overline{\underset{1}{W}}^{pq}, \qquad \underset{5}{W}^{pq} = \overline{\underset{2}{W}}^{pq}, \qquad \underset{6}{W}^{pq} = \overline{\underset{3}{W}}^{pq}.$$

Now, by writing the condition (18) for each of these bivectors and, taking into account that

$$\underset{\alpha}{\xi}^{\sigma} = \delta^{\sigma}_{\alpha}\,,$$

we can easily find

$$m_{ii} = -\underset{i}{\alpha}\,, \quad m_{ij} = 0\,, \quad n_{ii} = -\underset{i}{\beta}\,, \quad n_{ij} = 0\,, \quad (i = 1,2,3; \quad i \neq j);$$

and, therefore, *for the first type of T_4 we obtain the following canonical form of the matrix*:

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\underset{1}{\alpha} & & & -\underset{1}{\beta} & & \\ & -\underset{2}{\alpha} & & & -\underset{2}{\beta} & \\ & & -\underset{3}{\alpha} & & & -\underset{3}{\beta} \\ \hline -\underset{1}{\beta} & & & \underset{1}{\alpha} & & \\ & -\underset{2}{\beta} & & & \underset{2}{\alpha} & \\ & & -\underset{3}{\beta} & & & \underset{3}{\alpha} \end{array} \right\|, \tag{23}$$

the real parts of the stationary curvatures being related to each other in the following way:

$$\sum_1^3 \underset{s}{\alpha} = \varkappa\,, \tag{24}$$

whereas the imaginary parts obey the condition

$$\sum_1^3 \underset{s}{\beta} = 0 \tag{25}$$

due to the Ricci identity

$$R_{1423} + R_{1234} + R_{1342} = 0\,.$$

Let us now consider a T_4 with the characteristic of the second type: $[21, \overline{21}]$. As we have already seen (§ 2), one can use the principal directions and and invariant bundles of the matrices $P(K)$ and $\overline{P}(K)$ for choosing the principal directions and invariant bundles of the K-matrix. It follows that it is sufficient to consider, for example, the matrix $P(K)$ having the characteristic $[21]$.

With this characteristic, the tensor $P_{\alpha\beta} = -m_{\alpha\beta} + i\,n_{\alpha\beta}$ of the three-dimensional space has [6] one non-isotropic principal direction

$$(P_{\alpha\beta} - \underset{1}{K}\, g_{\alpha\beta})\, \underset{1}{W}{}^{\beta} = 0 \tag{26}$$

and one isotropic principal direction

$$(P_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\, \underset{2}{W}{}^{\beta} = 0\,, \tag{27}$$

the latter $(\underset{2}{W})$ being orthogonal to $\underset{1}{W}$. Additionally, there exists an isotropic vector $\underset{3}{W}{}^{\beta}$, orthogonal to $\underset{1}{W}{}^{\beta}$ and not to $\underset{2}{W}{}^{\beta}$, which, together with these latter vectors, form an invariant plane $\{\underset{2}{W}, \underset{3}{W}\}$ of the tensor $P_{\alpha\beta}$. This is expressed by

$$(P_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\, \underset{3}{W}{}^{\beta} = \sigma\, \underset{2}{W}_{\alpha}\,, \tag{28}$$

where σ is an arbitrary nonzero scalar, whose choice is up to us. This arbitrariness is the result of the fact that $\underset{2}{W}$, $\underset{3}{W}$, being isotropic, can be multiplied by any number without changing their norms.

Any principal direction or bundle of $P_{\alpha\beta}$ will define the corresponding principal directions and bundles of the tensor $R_{\alpha\beta}$; all of them being defined by the bivectors of the type (17).

Let the root $\underset{1}{K}$ corresponds to a simple elementary divisor $(K - \underset{1}{K})$ of the fields of the K-matrix and to a principal direction defined by the bivector $\underset{1}{W}{}^{\alpha}$. As this bivector is non-isotropic, we can apply to it all the above operations used in the previous case for $\underset{1}{W}{}^{\alpha}$. Therefore, we can find a real frame, with respect to which

$$\underset{1}{W}{}^{pq} = \underset{14}{\xi}{}^{pq} + i\, \underset{23}{\xi}{}^{pq}\,.$$

This frame is defined up to a rotation in the plane $\{\underset{2}{\overline{\xi}}\,\underset{3}{\overline{\xi}}\}$ and to a Lorentz rotation in the plane $\{\underset{1}{\overline{\xi}}\,\underset{4}{\overline{\xi}}\}$. As the bivectors $\underset{2}{W}{}^{pq}$ and $\underset{3}{W}{}^{pq}$

must be orthogonal to $\underset{1}{W}^{pq}$, they have the following form:

$$\underset{2}{W}^{pq} = \underset{2}{\mu}\,(\underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}) + \underset{2}{\nu}\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\,,$$

$$\underset{3}{W}^{pq} = \underset{3}{\mu}\,(\underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq}) + \underset{3}{\nu}\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\,.$$

The isotropy condition for these two bivectors results in

$$\underset{2}{\mu}^2 + \underset{2}{\nu}^2 = 0\,, \qquad \underset{3}{\mu}^2 + \underset{3}{\nu}^2 = 0\,,$$

that is,

$$\underset{2}{\nu} = e_1\, i\underset{2}{\mu}\,, \qquad \underset{3}{\nu} = e_2\, i\underset{3}{\mu}\,,$$

where e_1 and e_2 are equal to ± 1. Finally, using the fact that they cannot be orthogonal, we find that $e_1 = -\,e_2$. Therefore, we can put, for example,

$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq} + i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq}),$$

$$\underset{3}{W}^{pq} = \lambda\{\underset{24}{\xi}^{pq} + i\,\underset{31}{\xi}^{pq} - i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\},$$

where λ is an arbitrary scalar factor $\neq 0$.

Now we have only to write the conditions similar to (26), (27) and (28) for the tensor $R_{\alpha\beta}$, again, as in the previous case, taking into account that $\underset{\nu}{\xi}^{\alpha} = \delta^{\alpha}_{\nu}$. These conditions will have the form

$$(R_{\alpha\beta} - \underset{1}{K}\, g_{\alpha\beta})\,\underset{1}{W}^{\beta} = 0\,,$$

$$(R_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\,\underset{2}{W}^{\beta} = 0\,,$$

$$(R_{\alpha\beta} - \underset{2}{K}\, g_{\alpha\beta})\,\underset{3}{W}^{\beta} = \sigma g_{\alpha\beta}\,\underset{2}{W}^{\beta}.$$

The tensor $g_{\alpha\beta}$ is defined by the matrix (6). Assuming here $\alpha = 1, 2, \dots, 6$, we can readily find that *the matrix* $(R_{\alpha\beta})$ (11) *will be*

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\underset{1}{\alpha} & 0 & 0 & -\underset{1}{\beta} & 0 & 0 \\ 0 & -\underset{2}{\alpha}+\sigma & 0 & 0 & -\underset{2}{\beta} & \sigma \\ 0 & 0 & -\underset{2}{\alpha}-\sigma & 0 & \sigma & -\underset{2}{\beta} \\ \hline -\underset{1}{\beta} & 0 & 0 & \underset{1}{\alpha} & 0 & 0 \\ 0 & -\underset{2}{\beta} & \sigma & 0 & \underset{2}{\alpha}-\sigma & 0 \\ 0 & \sigma & -\underset{2}{\beta} & 0 & 0 & \underset{2}{\alpha}+\sigma \end{array} \right\|, \quad \sigma \neq 0\,. \tag{29}$$

Here σ can be arbitrary but $\neq 0$. As in the first case, $\underset{s}{\alpha}$ and $\underset{s}{\beta}$ are related to each other through

$$\underset{1}{\alpha} + 2\underset{2}{\alpha} = \varkappa\,, \qquad \underset{1}{\beta} + 2\underset{2}{\beta} = 0\,. \tag{30}$$

The frame is determined up to a rotation in the plane $\{\underset{2}{\bar{\xi}}\,\underset{3}{\bar{\xi}}\}$ and a Lorentz rotation in the plane $\{\underset{1}{\bar{\xi}}\,\underset{4}{\bar{\xi}}\}$.

We have to consider now the third type with the charcteristic $[\,3, \bar{3}\,]$. For this characteristic [6], the tensor $R_{\alpha\beta}$ will have only one principal isotropic direction $\underset{1}{W}^{\beta}$ and, additionally, two more vectors $\underset{2}{W}^{\beta}$ and $\underset{3}{W}^{\beta}$ with the properties

$$\left.\begin{aligned} &(R_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta})\underset{1}{W}^{\beta} = 0 \\ &(R_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta})\underset{2}{W}^{\beta} = \sigma\,\delta_{\alpha\beta}\,\underset{1}{W}^{\beta} \\ &(R_{\alpha\beta} - \underset{1}{K}\,\delta_{\alpha\beta})\underset{3}{W}^{\beta} = \tau\,\delta_{\alpha\beta}\,\underset{2}{W}^{\beta} \end{aligned}\right\}, \tag{31}$$

where σ and τ are arbitrary numbers $\neq 0$. The vector $\underset{2}{W}^{\alpha}$ is non-isotropic, whereas $\underset{3}{W}^{\alpha}$ is isotropic. Besides that, $\underset{1}{W}^{\alpha}$ is orthogonal to $\underset{2}{W}^{\alpha}$ and not orthogonal to $\underset{3}{W}^{\alpha}$; while the vector $\underset{2}{W}^{\alpha}$ being orthogonal to $\underset{3}{W}^{\alpha}$.

Since $\underset{2}{W}^{pq}$ is not an isotropic bivector, then, similarly to the previous two cases, we can write this vector as

$$\underset{2}{W}^{pq} = \underset{24}{\xi}^{pq} + i\underset{31}{\xi}^{pq}$$

by choosing an appropriate frame (with two degrees of freedom). Then, by taking into account the above conditions for orthogonality and isotropy, we shall get the following expressions for the bivectors $\underset{1}{W}$ and $\underset{2}{W}$:

$$\underset{1}{W}^{pq} = \underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq} + i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\,,$$

$$\underset{2}{W}^{pq} = \lambda\{\underset{14}{\xi}^{pq} + i\,\underset{23}{\xi}^{pq} - i\,(\underset{34}{\xi}^{pq} + i\,\underset{12}{\xi}^{pq})\}\,,$$

where λ is an arbitrary number $\neq 0$. The further study is made following the same scheme as for the previous characteristic types: we should write the conditions (30) for $R_{\alpha\beta}$, fixing the facts that $\underset{1}{W}^{\alpha}$ is the vector of the principal direction (in the bivector space) and that the vectors $\underset{1}{W}^{\alpha}$, $\underset{2}{W}^{\alpha}$, $\underset{3}{W}^{\alpha}$ determine the invariant bundle of the tensor $R_{\alpha\beta}$.

These conditions are as follows:

$$\left.\begin{aligned} &(R_{\alpha\beta} - K\,g_{\alpha\beta})\underset{1}{W}^{\beta} = 0 \\ &(R_{\alpha\beta} - K\,g_{\alpha\beta})\underset{2}{W}^{\beta} = \sigma\,g_{\alpha\beta}\underset{1}{W}^{\beta} \\ &(R_{\alpha\beta} - K\,g_{\alpha\beta})\underset{3}{W}^{\beta} = \tau\,g_{\alpha\beta}\underset{2}{W}^{\beta} \end{aligned}\right\}, \tag{32}$$

where σ and τ are non-zero numbers.

Considering that at any given point of T_4 the bivector $\underset{\sigma}{W}^{pq}$ corresponds to the vector $\underset{nt}{W}^{pq} \longrightarrow \underset{\sigma}{W}^{\alpha}$ in a local bivector metric space and taking into account that for the coordinate frame

$$\underset{nt}{\xi}^{pq} \longrightarrow \underset{\sigma}{\xi}^{\alpha} = \delta^{\alpha}_{\sigma},$$

it is not difficult to check that the system of equations (32) is reduced to the following nine independent equations:

$$\begin{aligned} &m_{11} + i n_{11} + i m_{13} - n_{13} = -K\,, \\ &m_{12} + i n_{12} + i m_{23} - n_{23} = 0\,, \\ &m_{13} + i n_{13} + i m_{33} - n_{33} = -\,iK\,, \\ &m_{12} + i n_{12} = -\,\sigma\,, \\ &m_{22} + i n_{22} = -K\,, \\ &m_{23} + i n_{23} = -\,i\sigma\,, \\ &m_{11} + i n_{11} - i m_{13} + n_{13} = -K\,, \\ &m_{12} + i n_{12} - i m_{23} + n_{23} = -\,\tau\,, \\ &m_{13} + i n_{13} - i m_{33} + n_{33} = iK\,, \end{aligned}$$

where $K = \alpha + i\beta$ is one of the two 3-fold roots of the secular equation

$$|\,R_{\alpha\beta} - K\,g_{\alpha\beta}\,| = 0\,,$$

and the numbers σ and τ are arbitrary but not equal to zero. This arbitrariness ensues from the arbitrariness of λ and is due to the isotropy of the vectors $\underset{1}{W}^{\alpha}$, $\underset{3}{W}^{\alpha}$. For instance, one can assume that σ and τ are real numbers.

By solving this system and also taking into account the conditions

$$\sum_{s=1}^{3} e_s m_{ss} = \varkappa\,, \qquad \sum_{s=1}^{3} e_s n_{ss} = 0\,,$$

one can check that $\tau = 2\sigma$, $\beta = 0$, $\alpha = \frac{\varkappa}{3}$, and the matrix $\| R_{\alpha\beta} \|$ takes the following form:

$$(R_{\alpha\beta}) = \left\| \begin{array}{ccc|ccc} -\frac{\varkappa}{3} & -\sigma & 0 & 0 & 0 & 0 \\ -\sigma & -\frac{\varkappa}{3} & 0 & 0 & 0 & -\sigma \\ 0 & 0 & -\frac{\varkappa}{3} & 0 & -\sigma & 0 \\ \hline 0 & 0 & 0 & \frac{\varkappa}{3} & \sigma & 0 \\ 0 & 0 & -\sigma & \sigma & \frac{\varkappa}{3} & 0 \\ 0 & -\sigma & 0 & 0 & 0 & \frac{\varkappa}{3} \end{array} \right\|, \tag{33}$$

where σ is an arbitrary non-zero number; the frame is determined up to a rotation in the two-dimensional plane $\{ \underset{1}{\xi} \underset{3}{\xi} \}$ and a Lorentz rotation in the plane $\{ \underset{2}{\xi} \underset{4}{\xi} \}$.

As the final result, we have the following theorem.

Theorem. *There exist three fundamentally distinct types of gravitational fields:*

The 1st type, with the characteristic of the K-matrix of the simple type $[\,1\,1\,1\,, \overline{1}\,\overline{1}\,\overline{1}\,]$, for which a real orthogonal frame is uniquely defined at any point of T_4, and with respect to which the matrix $\| R_{\alpha\beta} \|$ has the form (23) under the conditions (24) and (25).

The 2nd type, with the characteristic of a non-simple type $[\,2\,1\,, \overline{2}\,\overline{1}\,]$, for which the frame is defined having two degrees of freedom, and the matrix $\| R_{\alpha\beta} \|$ has the form (29) under the conditions (30).

The 3rd type has also the characteristic of a non-simple type $[\,3,\, 3\,]$; its frame has two degrees of freedom, and its matrix $\| R_{\alpha\beta} \|$ has the form (33).

Here the overlined numbers in the characteristics denote the power indices of those elementary divisors, whose bases are complex-conjugate to the bases corresponding to the numbers without overlining.

The three indicated types obviously admit some further more detailed classification. For example, one can distinguish the cases of multiple or real roots, as had been already done by the author earlier. This result, which I have obtained in 1950, was first published in 1951 in [1]. There is an ambiguity in the formulation given in that paper. The proof of the theorem from § 2 was also provided by A. P. Norden in 1952 (which was not published), whose starting point was from his study of bi-affine spaces. The proof given here is the third one and it is probably the simplest one.

As for the study carried out in § 3 (i.e., the determination of the canonical form of the matrix $(R_{\alpha\beta})$ for the orthogonal non-holonomic frame), we have to make the following note. At first thought, one might expect to approach this task in the following way: since the characteristic of the matrix $\| R_{\alpha\beta} - K g_{\alpha\beta} \|$ is known, it seems to be possible to write directly the canonical form of this matrix base on the general algebraic theory [6]. However, this cannot be done because the coefficients of admissible linear real transformations can be taken only in the form

$$A_{\alpha}^{\alpha'} = 2A_{ij}^{[j'j']},$$

where $A_i^{i'} = \left(\frac{\partial x^{i'}}{\partial x^i}\right)_P$ are the coefficients of some real orthogonal transformation at a given point P of the manifold T_4. That is, we can only use the transformations belonging to a subgroup of the group of all real orthogonal transformations in a 6-dimensional space.

This fact, which requires the arguments of § 3, is in our case obvious; it is a specific application of a more general theorem proved by G. B. Gurevich [7].

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