On the Permissible Numerical Value of the Curvature of Space

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Abstract: This is a translation of Schwarzschild's pioneering presentation where he pondered upon the possible non-Euclidean structure of space and gave a lower limit for the measurable radius of curvature of space as 4,000,000 astronomical units (supposing the space to be hyperbolic) or 100,000,000 astronomical units (elliptic space). The paper was originally published as: Schwarzschild K. Über das zulässige Krümmungsmaaß des Raumes. *Vierteljahrsschrift der Astronomische Gesellschaft*, 1900, Bd. 35, S. 337–347. Translated into English in 2008 by Dmitri Rabounski. The translator thanks Ulrich Neumann, Germany, for a copy of the Schwarzschild manuscript in German, and also Stephen J. Crothers, Australia, for assistance.

Permitting myself to call your attention for this presentation, which has neither practical purpose nor mathematical meaning, I should be excused due to the theme of the presentation itself. This theme is obviously very attractive to most of you due to the fact that it is related to the expansion of our views to boundaries far away from our everyday experience, and opens beautiful horizons for possible experiments in the future. The fact that all these lead us to the failure of numerous traditional views which are most hard rooted in the heads of astronomers, is just an advantage of this new theme from the view of everyone who believes in the relativity of our knowledge.

This talk is on the permissibility of curved space. You all know that in the 19th century along with the Euclidean geometry numerous other non-Euclidean geometrical systems were developed, which were headed by the geometrical systems of so-called spherical space and of so-called pseudo-spherical space (we will deal mainly with these two systems here). It is possible to develop in detail a picture of what would be observed in a spherically curved space or a pseudo-spherically curved space. I however limit myself by only a reference to Helmholtz' paper *The Origin and Meaning of the Geometrical Axioms**. Here we

*Hermann von Helmholtz. Über den Ursprung und die Bedeutung der geometrischen Axiome. *Vortrag gehalten im Docentenverein zu Heidelberg*, 1870, Universitätsbibliothek Heidelberg. Published in English as: Hermann Helmholtz. The origin and meaning of geometrical axioms. *Mind*, July 1876, vol. 1, no. 3, pages 301–321. — Editor's comment. D.R.

are come into a fairyland of geometry, which is especially beautiful due to the fact that it may relate to our real world, and finally we are unsure in the impossibility of it. Here we consider how wide the boundaries of this fairyland can be expanded, *what is the largest numerical value of the permissible curvature of space, what is the smallest radius of the space curvature.*

One usually answers this question unsatisfactorily, at least unsatisfactorily from the viewpoint of an astronomer. In Euclidean geometry the sum of the angles in a triangle is $2d$; while in the case of the non-Euclidean geometry the larger the triangle we are considering, the more this sum differs from $2d$. One may point out that even in the case of the largest of the measured triangles (the apex of such a triangle is a star, while the base is drawn by the diameter of the Earth's orbit) the sum of the angles in each of them wasn't found to be different from $2d$. Hence the curvature of space should be negligible. In such an answer people overlook just one circumstance. They don't take into account the circumstance that the angle at the star isn't a subject of measurement, but is obtained as a calculation resulting from the theorems of Euclidean geometry, the correctness of which will be the subject of our consideration here. Besides, an astronomer shouldn't be satisfied by the note, according to which he should neglect the curvature of space in the scale of the nearest stars, whose parallax is accessible to measurement; to obtain a picture of the interior of the world of stars, he should take into account the distances to even the weakest stars, which are far relative to us.

I begin consideration of this problem from the point of view which gives the possibility of talking about the theoretical meaning of non-Euclidean geometry. In order to measure the positions of three vertices of a triangle, we will employ the light beams coming from one of the vertices. The lengths of the sides a, b, c of this triangle will be measured according to the duration of time required for the light beam to travel along the lengths, while the angles α, β, γ will be measured by a regular astronomical instrument. Our everyday experience manifests in plane trigonometry, true on all triangles within the precision of measurement. Suppose that the regular trigonometry is not absolutely precise and that in reality the sides and the angles of triangles are connected by the following relations

$$\sin\alpha : \sin\beta : \sin\gamma = \sin\frac{a}{R} : \sin\frac{b}{R} : \sin\frac{c}{R}\,, \qquad (a)$$

$$\cos\frac{c}{R} = \cos\frac{a}{R}\cdot\cos\frac{b}{R} + \sin\frac{a}{R}\cdot\sin\frac{b}{R}\cdot\cos\gamma\,. \qquad (b)$$

Here R is a very large interval we will refer to as the *curvature radius of space*, by which we mean no close analogy to the curvature radius known in the geometry of two dimensions. The aforementioned formulae coincide with the main formulae of spherical trigonometry which, as well known, transform into the regular trigonometric formulae in the case where the sides of the triangle are small relative to the radius R of the sphere. However taking R sufficiently large, the sides of any triangle we are measuring become small relative to R. Therefore, by increasing R we can always arrive at a case where the formulae (a) and (b) meet the regular trigonometric formulae within the measurement precision. In order words, it is enough to increase R to reduce the formulae (a) and (b) into coincidence with our everyday experience.

Here we don't consider a purely mathematical problem on the grounds of acceptance of formulae (a) and (b) for any triangle, without internal contradiction. As we know this question has been answered positively. Besides, as shown by research, the requirement that spherical trigonometry be applicable to all triangles in a space provides no exact information about coherence of the space. Among the possible forms of space which permit spherical trigonometry, the simplest and most well-known are the so-called *spherical space* and the *elliptic space*. The following common properties are attributed to a spherical space and an elliptic space: such a space is finite, the volume of it is finite as well and is dependent on the curvature radius. By following a path in such a space, we arrive at the initial point. Relations given in a plane of such a space are absolutely the same as those on the surface of a sphere according to usual views. Besides that, a plane located in a curved space is determined, as usual and everywhere, by all straight lines — all beams of light which pass through two crossed light beams. Any straight line in a plane of a such curved space is similar to a great circle on the surface of a sphere. For two parallel straight lines, i.e. two straight lines crossing a third straight line at equal angles (two right angles, for instance), these straight lines are similar to two meridians crossing the equator at right angles. Similarly for the crossing meridians at the point of the pole, the straight lines cross each other in a curved space at the distance $\frac{\pi}{2}R$ in a curved space. One may say that, concerning a plane of a curved space, two parallel straight lines should cross each other twice like two great circles on a sphere. This hypothesis lies at the foundation of spherical space. However it is possible that two parallel straight lines cross each other only once; this assumption leads us to "elliptic" space. It is possible to map a plane in a curved space onto a usual spherical surface in such a way that each point of the plane conjugated, not only with the radius, but also the diameter and, hence, two diametrically opposite points of the spherical surface. Therefore, if taking great circles passing through a point of a spherical surface, and crossing each other at the diametrically opposite point, the incoming point and the point diametrically opposite to it are similar to a single point of a plane in a curved space, where the respective straight lines cross each other. From this we conclude that we, travelling by way of the length πR (not $2\pi R$), arrive at the initial point and, at the same time, the maximum distance between two points in such a space is $\frac{\pi}{2} R$. Similarly we study *elliptic space*, which is the simplest of spherical trigonometry spaces. (In talking above about spherical space instead elliptic space, we merely used the more common and usual term.)

But first we should mention another very simple generalization of non-Euclidean geometry. If in (a) and (b) we replace R with an imaginary quantity iR, we obtain

$$\sin\alpha : \sin\beta : \sin\gamma = \mathrm{Sin}\,\frac{a}{R} : \mathrm{Sin}\,\frac{b}{R} : \mathrm{Sin}\,\frac{c}{R}\,, \qquad (a')$$

$$\mathrm{Cos}\,\frac{c}{R} = \mathrm{Cos}\,\frac{a}{R}\cdot\mathrm{Cos}\,\frac{b}{R} + \mathrm{Sin}\,\frac{a}{R}\cdot\mathrm{Sin}\,\frac{b}{R}\cdot\cos\gamma\,, \qquad (b')$$

where capital letters denote hyperbolic functions. These equalities transform into the equalities of plane trigonometry with the increase of R. There are various spatial forms wherein the special trigonometry based on formulae (a') and (b') are true. The simplest of these spatial forms is the so-called "*pseudo-Riemannian*" or "*hyperbolic*" *space*. Such a space is infinite: therein each point is crossed by a couple of straight lines without intersecting another given straight line. The geometry on any of the planes of such a space is analogous to the geometry on a so-called pseudo-sphere, which is constant negative curvature surface.

Now we turn our attention to the problem of how to determine the *parallax* in the cases of both elliptic and hyperbolic spaces. Any of the definitions of parallax can be reduced to the following: given two times of observation separated by a half year duration, we measure the angle created at the Earth by two straight lines which connect it with two stars we observe. Assume, for simplicity, that one of these two stars, S_1, is positioned exactly in the continuation of the diameter of the Earth's orbit, while the other star, S_2, is positioned in the line which is approximately orthogonal to this direction. Denoting E_1 and E_2 as the positions of the Earth at the times of observation ($E_1E_2 = r$ is the diameter of the Earth's orbit), the observations give both angles $S_1E_1S_2 = \alpha$ and $S_1E_2S_2 = \beta$. The quantity $p = \frac{\alpha-\beta}{2}$ is known as the parallax of the star S_2. The problem is as follows: having the elements α, β, $2r$, how to calculate the distances $E_1S_2 = a$ and $E_2S_2 = b$ from the star S_2 to both locations of the Earth in the cases of spherical trigonometry and pseudo-spherical trigonometry. Because the straight line directed at S_2 should be approximately orthogonal to $E_2E_1S_1$, we can assume $a = b = d$ where d is the distance from the star. We take into account that fact that the parallax p is a very small angle, and the curvature radius of the space should be undoubtedly much larger than the diameter of the Earth's orbit. With these we easily obtain the following formulae for the distance in the case of an elliptic space

$$\operatorname{cotg}\frac{d}{R} = \frac{R}{r}\,p \quad \text{or} \quad \sin\frac{d}{R} = \frac{1}{\sqrt{p^2R^2 + r^2}}\,, \tag{c}$$

and that in a hyperbolic space

$$\operatorname{Cotg}\frac{d}{R} = \frac{R}{r}\,p \quad \text{or} \quad \operatorname{Sin}\frac{d}{R} = \frac{1}{\sqrt{p^2R^2 - r^2}}\,. \tag{c'}$$

The last of these formulae leads to a conclusion concerning *hyperbolic space*. Naturally, given each real distance d, the inequality $pR > r$ should hold. Therefore there is a minimum parallax, which is $p = \frac{r}{R}$, that should be observed for even very distant stars. On the other hand we know of many stars which don't have a parallax of even $0.05''$. Hence the numerical value of the minimal parallax should be lesser than $0.05''$. We obtain the lower limit of the curvature radius of the hyperbolic space

$$R > \frac{r}{\text{arc}\,.050''} \quad \text{that is} \quad R > 4,000,000 \text{ radii of the Earth's orbit.}$$

According to this the curvature of the hyperbolic space is so small that it doesn't manifest in any measurements on the scale of the planetary system. Besides, because any hyperbolic space is infinite, as is any Euclidean space, it is impossible to find unusual phenomena by observation of stars in the sky.

Before consideration of *elliptic space*, I remark that it was recently shown by Prof. Seeliger that the most accurate representation of our stellar system, on the basis of the observational data, concludes that all stars we observe (the number of the stars is no greater than 40 million) are located inside the space, the diameter of which is a few hundred million times larger than the radius of the Earth's orbit, beyond which a large and approximately empty space begins. This concept bears somewhat comfortably upon our minds, because according to it the complete study of the limited stellar system is an special stage in the evolution of our knowledge about the world. But this comfort and satisfaction would be much more effective if we imagined the space enclosed in itself, in a finite and complete manner, or approximately filled with this stellar system. Naturally, if so, we could reach a stage when the space has been studied completely, like the surface of the Earth has been studied, so that any macroscopic studies of the space have ceased being subordinate to microscopic studies. These very advanced studies may explain, in my view, that strong interest that attracts us to the hypothesis of elliptic space.

Now we look at the results of *calculation of the parallax in the elliptic space*. Employing the aforementioned formula

$$\operatorname{cotg}\frac{d}{R} = \frac{R}{r}\,p\,,$$

we can obtain, concerning any measured parallax p of a star, a specific real numerical value of the distance d to the star at *any numerical value of the curvature radius* R. Thus we see that it would be erroneous to think that the limit of R was found proceeding from only our measurements of the stellar parallaxes. According to these measurements, it would be possible that the space was so strongly curved that, travelling along a path equal to approximately 1,000 distances from the Earth to the Sun (i.e. the distance travelled by light during a few days), we would arrived at the initial point of our journey. Therefore, not purely metric reasons but physical reasons lead us to a conclude that the curvature radius is much larger than that suggested.

A very small curvature radius would lead to the metric inconsistencies in the planetary system. Because we further find a greater upper limit of it, it is enough to say that, in the case of the curvature radius equal to 30,000 radii of the Earth's orbit, it produces an imperceptible effect even in triangles which are as large as the distance to the orbit of Neptune. This radius of the space curvature corresponds to the length which is no larger than 1/10 part of the distance to the nearest stars.

So, assume $R = 30,000$ radii of the Earth's orbit. According to formula (c), we calculate the distance to the stars at different numerical values of the parallax. We obtain

$$\text{for } p = 1.0'' \qquad 0.908\,\frac{R\cdot\pi}{2} = 42,800 \text{ radii of the Earth's orbit,}$$

$$\text{for } p = 0.1'' \qquad 0.991\,\frac{R\cdot\pi}{2} = 46,700 \text{ radii of the Earth's orbit,}$$

$$\text{for } p = 0.0'' \qquad 1.000\,\frac{R\cdot\pi}{2} = 47,100 \text{ radii of the Earth's orbit.}$$

It is easy to see that we have arrived at quite ridiculous results. There are maybe a hundred stars whose parallax is $p > 0.1''$. Thus these hundred stars should be scattered at distances between one another no larger than 46,700 radii of the Earth's orbit, while the rest of the space at only 400 radii of the Earth's orbit is reserved for the remaining millions of stars. In such a case the Sun would be located in a space of exceptionally small stellar density, while everywhere beyond a certain distance from it there is an exceptionally large density of stars. To show this density of stars, I calculated the volume of the space limited by 46,700 radii of the Earth's orbit, and also the volume of the remaining part of the space, then I calculated the average distance between two stars assuming that there is exactly 100 million stars in total. I found that in the approximately empty space near the Sun the average distance between two stars is about 15,000 radii of the Earth's orbit, while in the high density inhabited rest of the space the average distance is only 40 radii of the Earth's orbit. Of course, it is impossible to accept such a calculation result that stars are so close to each other; otherwise it would be found in the physical interactions among the stars. It follows that the supposed curvature radius of 30,000 radii of the Earth's orbit is too small.

It is clear that by increasing R we may overcome all these difficulties, because they all vanish at $R = \infty$ (this is an obvious assumption). It is enough to take R so large that those 100 million stars with parallaxes less than $0.1''$ we assumed inhabit the space, which is a million times larger than the space inhabited by 100 million stars with parallaxes bigger than $0.1''$. Simple algebra shows that this takes place for

$$R = 160,000,000 \text{ radii of the Earth's orbit.}$$

In the case of a similar radius of the space curvature, light would travel around the whole space, along the path πR, in 8,000 years. However the size of the respective elliptic space is approximate the same as that suggested by Seeliger for the finite system of resting stars, not yet so large a size as that of the stellar system known according to the usual bounds. One could suggest R to be two or three times less than the above, but even such a reduction of R doesn't lead to the suggested abnormal emptiness of stars in the neighbourhood of the Sun and their high density at large distances from it.

Thus we arrive at the conclusion that the assumption, according to which R is equal to approximately 100,000,000 radii of the Earth's orbit, doesn't contradict the observational data. In the case of such a numerical value of R the whole finite space is homogeneous, filled with the observable stars.

One more fact should also be noted here. Given an elliptic space, any light beam arrives back at its initial point after travelling across the whole space. So light beams emitted into such a space from the opposite (invisible to us) side of the Sun should travel across the space then also meet the Earth, then create an anti-image of the Sun from the opposite side of the real image of it. This anti-image shouldn't fade with respect to the real image of the Sun, because light beams compress upon returning to the initial point of travel, becoming such ones as they travel in the least direct way from the source of light. But due to that fact that such an anti-image of the Sun was never observed we are enforced to suppose that light, travelling across the whole space, experiences absorption which is so strong that the anti-image is invisible. This supposition is true if supposing the absorption to be approximately 40 stellar magnitudes. There is no facts against the supposition of such a numerical value of the absorption, which seems small compared to the scale of the Earth.

In conclusion: *it is possible to imagine, with no contradiction of the experimental data, that the world is closed within a hyperbolic (pseudo-spherical) space, the curvature radius of which is larger than 4,000,000 radii of the Earth's orbit, or, alternatively — within an elliptic space, the curvature radius of which is larger than 100,000,000 radii of the Earth's orbit. In addition, in the second case, it should be supposed an absorption of light equal to 40 stellar magnitudes per around space travel.*

Now we should limit ourselves by these. At least, I see no other way to make a principal step in this direction with use of the contemporary methods of research, i.e. how to prove that the volume of the space is so large with respect to the volume of the stellar system we observe, or that the space has a really positive or negative curvature. On the other hand, I can provide some considerations which, despite providing no definite solution, may bring us to a specific preferential numerical value of R within the aforementioned scale of the values.

It is well known that astronomers, in their study of the distribution of stars in space, proceed from the simplest possible rational hypotheses about the average luminosities of stars, then they distribute the stars at different distances from the Sun by such methods that arrive at the numbers of stars of each stellar magnitude obtained in astronomical observations. Such research — the main result of which was mentioned above — was already produced by Prof. Seeliger. It could be produced in the same way in the cases of a pseudo-spherical space or of an elliptic space. I have calculated, in the cases of both spatial forms, the dependence of the number of stars from their stellar magnitude in the approximation that the luminosities of all stars are the same, and also that the density of the stellar population in all regions of the space is uniform. I have found, using the same physical assumptions, that the number of stars increases with the increase of their luminosity more slowly in the pseudo-spherical space in contrast to that in the Euclidean space, while in an elliptic space it increases faster than in the Euclidean space. In the real situation, as is well known, the number of stars increases with their luminosity slower than expected on the basis of the simple hypotheses about the Euclidean space. Proceeding from this fact, it could be concluded that the pseudo-spherical space is real. But, of course, no serious meaning can be attributed to these speculations, because the hypotheses of the equal luminosities and the equal density of stars take, as probable, no place in the real situation. However, as I have already said, this theory could be developed in the case of a curved space on the same bases used by Prof. Seeliger, who developed the theory in Euclidean space. Comparing the conclusion with the observational data, one could say then that the simplest picture of the distribution of stars is obtained on the assumption that space has a non-zero curvature. Of course, it is impossible to expect that a definite and final answer will be obtained here. We therefore have to accept that sad fact that there is little hope for a solid proof to the finitude of space.

Appendix. In the above, of all the spatial forms where "free motion of solid bodies" is possible, only the main types were considered (as has been noted by F. Klein). In order to finalize this theme, the other spaces which have this property should be compared to the astronomical data. I would exclude from consideration "spherical space" and other so-called "double-spaces", where all light emitted from a point travels to another point, collecting all the light anew. This is because we have no reason for introduction of such a complicated hypothesis. Therefore we have to settle for the so-called "*simple Clifford-Klein spatial forms*".

Of all these spatial forms, special is the one which amplifies the fact that the acceptance of Euclidean geometry is not equivalent, as one usually thinks, to the indefiniteness of the space. Imagine that we, after greatly enhanced astronomical data, found that our universe consists of countless copies of our Milky Way, that the infinite space can be split into many cubes, each of which contains a stellar system that is absolutely equivalent to the system of our Milky Way. Do we really stop at the assumption of an infinite number of identical copies of the same world-entity? To understand the absurdity of this, think about just one sequel: in such a case we ourselves, the observing objects, should exist in an infinite number of manifestations. We would better go to the assumption that these copies are only imaginary images, while the real space permits such coherences due to which we, being left the cube from one side and travelling always along a straight path, arrive at the cube from its opposite side. Such a space as we have supposed is nothing but the simplest of the Clifford-Klein spatial forms: a finite space of Euclidean geometry. It is easy to see the sole condition which should be attributed to such a Clifford-Klein space: because as yet nothing has been found concerning the (imaginary) copies of the system of the Milky Way, the volume of the space should be bigger than the volume we attribute to the Milky Way on the basis of the theorems of Euclidean geometry.

About the other simple Clifford-Klein spatial forms, we limit ourselves by only a few words, due to that fact that these spaces aren't sufficiently studied as yet. All these forms are obtained in analogous way by the identical imaginary copying of the same world-entity in a Euclidean space, in an elliptic space, or in a hyperbolic space. Experimental data lead us, again, to the condition according to which the volume of any such spaces should be bigger than the volume of the stellar system we observe.

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