On the Relativistic Theory of an Anisotropic Inhomogeneous Universe

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Abstract: Here the General Theory of Relativity is expounded from the point of view of space-time as a continuous medium, and the mathematical apparatus for calculation of physically observable quantities (the theory of chronometric invariants) is constructed. Then this mathematical apparatus is applied to set up the basics of the theory of an inhomogeneous anisotropic universe, which profitably contrasts the self-limited theories of a homogeneous universe (most commonly used in modern relativistic cosmology). Owing to such an extension of the relativistic cosmology, we determine the whole range of cosmological models (scenarios of evolution) which could be theoretically conceivable in the space-time of the General Theory of Relativity. Translated from the original Russian manuscript of 1957, in 2008 by Dmitri Rabounski.

§1. The question "is the Universe homogeneous and isotropic, or not" is connected with the question about the scale of the Universe. Let l be a length which is in the order of the upper limit of the space regions meant, by us, to be infinite small. Then $L \gg l$ is a length, which is in the order of the size of the whole region of space we observe. As obvious, in connexion to the question about the scale, two different understandings about homogeneity and isotropy are possible. In other words, two questions can be asked: 1) are the conditions of homogeneity and isotropy satisfied at the numerical values of l and L, assumed by us; 2) is there a large enough l that, under any $L \gg l$, the conditions of homogeneity and isotropy are satisfied.

In comparing the theory to observations, the first of the above understandings of homogeneity and isotropy plays a rôle. In such a case the numerical values of l and L should be determined at least in the order of these values. In consideration of questions such as those related to the infinity of space, the second understanding of homogeneity and isotropy is important. Observational data give no direct answer to the question about homogeneity and isotropy of the Universe with respect to the second meaning. I don't provide the references to the observational data here. On the other hand, much information about the distribution of masses, provided for instance by Ambarzumian in his presentation [1], allows us to be sure in the fact that, at any $l \ll L$, the Universe is inhomogeneous, in the first meaning of this term. Ambarzumian was absolutely right in his note that the Metagalactic redshift should be interpret as the Doppler effect, and, when considering the scale of the Metagalaxy, we should take into account the effects of the General Theory of Relativity. Thus a relativistic theory of an inhomogeneous anisotropic universe — a theory, the results of which would be able to be compared to the observational data, and which, generally speaking, gives a model of the whole Universe — should be our task.

As will be shown in §10, inhomogeneity leads to anisotropy.* On the other hand, at least some factors of anisotropy bear a tendency to decrease with the expansion of the Metagalaxy (see §13). So the anisotropy, being weak in the current epoch, was probable a valuable factor which played an important rôle billions of years ago.

So a relativistic theory of an inhomogeneous anisotropic universe is our actual task. The necessity of such a theory was pointed out, aside for the special studies on this theme, in [6, 7] and also in [8]. Below, only a few problems related to the formal mathematical basics of this theory will be considered. It should be noted that the basic equations and the deduced equations of our theory do not depend on homogeneity and isotropy of the Universe in the second meaning: the equations are independent of the numerical values of l and L.

Assume that matter on the scale we are considering is a continuous medium, which moves laminarely in common with a continuous field of sub-luminal velocities. So there are coordinate frames which everywhere accompany the medium. On the other hand, all that will be said in §3–§8 does not depend on these assumptions. Of course, in the modern epoch of observation, the most interesting case is such a scale of consideration where the "molecules" of this medium are galactic clusters. Suppose also that, in the scale we are considering, the thermodynamical terms are meaningful, and the laws of relativistic thermodynamics hold. We also assume that Einstein's equations

$$G_{\mu\nu} = -\varkappa\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T\right) + \Lambda\, g_{\mu\nu} \tag{1}$$

are true everywhere in the four-dimensional region we are considering. Here in the equations $G_{\mu\nu}$ is the contracted world-tensor of the curvature, $g_{\mu\nu}$ is the metric world-tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $T = T^{\alpha}_{\alpha}$, $\varkappa$ is Einstein's constant of gravitation ($\varkappa = 8\pi\gamma/c^2$, where γ is Newton's constant of gravitation and c is the fundamental velocity), while Λ is the cosmological constant. We keep the cosmological constant

*In addition to it, there are observational data about anisotropy of the redshift in a small region of the Metagalaxy near us [2–4]. Of course, this information should be checked on the basis of the newest observational data [5].

in the equations, because we like to have a possibility to compare our results to those known in the literature.

§2. We always mean spatial (three-dimensional) homogeneity and isotropy, not four-dimensional world-quantities. The presence or the absence of spatial homogeneity and spatial isotropy depends on the frame of reference. For instance, as obvious, the isotropy can be attributed to only those reference frames, which accompany continuous matter and masses, because both the flow of matter and moving masses break the isotropy. On the one hand, the question about the presence of homogeneity and isotropy can be set up as the question about the possibility of such reference frames, where the said homogeneity and isotropy take a place. We all know the homogeneous isotropic relativistic models. In such a model, a frame consisting of the four coordinates can be chosen, wherein

$$\left.\begin{aligned} ds^2 = c^2dt^2 - R^2\,\frac{d\xi^2 + d\eta^2 + d\zeta^2}{\left[1 + \frac{k}{4}\left(\xi^2 + \eta^2 + \zeta^2\right)\right]^2} \\ R = R(t), \qquad k = 0, \pm 1 \end{aligned}\right\}. \tag{2}$$

Such a reference frame can be the necessary and sufficient indication of homogeneity and isotropy in cosmology. On the other hand, the question about the presence of homogeneity and isotropy can be set up in a frame of the accompanying coordinates. Such a statement of this problem will be realized in the next Paragraphs.

The theory of an inhomogeneous anisotropic universe has two main directions, which are characterized as follows: a) the search for exact particular solutions of the equations of gravitation, and the consideration of such models which bear the properties of symmetry; b) as common as possible, the qualitative study of the behaviour (evolution) of matter and the metric under different physical assumptions.

The models, which are spherically symmetric under the vanishing of the pressure, viscosity, and the flow of energy, the models with a spherically symmetric distribution of matter concentrated in a centre (core), and the models filled with a limited spherical distribution of matter were studied by McVittie [9], Tolman [10, 12], Datt [11], Oppenheimer and Volkoff [13], Oppenheimer and Snyder [14], Järnefelt [15, 16], Einstein and Strauss [17], Bondi [18], Omer [19], Just [20]. The models, which are axially symmetric and rotating, were considered in the studies of Kobushkin [21] and Gödel [22]. There are main studies produced in the research direction a), or connected to it.

Among the studies produced in the research direction b), McCrea's study [23] remains aloof, where the problem of the observable properties of an inhomogeneous anisotropic universe was considered. The behaviour (evolution) of matter and the metric in such a universe was qualitatively considered in studies of mine [24, 26], Raychaudhury [25] and Komar [27]. In the study [24] I introduced chronometrically invariant quantities (using another terminology), and considered applications of them in the General Theory of Relativity to the problem we are now interested in, in the framework of the particular conditions, where the flow of energy, viscosity, pressure, and, hence, the power field were neglected. A few years later, Raychaudhury [25] considered particular aspects of the same problem in the case where $\Lambda = 0$, with the neglection of the same factors. The quantities and equations derived by him, and also his conclusions [25] are the same as that which was found by me earlier [24]. Raychaudhury however did not introduce chronometrically invariant quantities, and used the incorrect definition (12) of the observable spatial metric instead of the correct formula (7) given below. As a result, his equations, generally speaking, don't possess a direct physical interpretation in the framework of the considered problem. Meanwhile, using [24, 26] one can show that his results concerning the effects produced by, in our terminology, the absolute rotation and the anisotropy of the deformations in a) the behaviour of the changes of a space volume and b) the scale of time are correct in the considered case. His results in the research direction a) repeated some results obtained earlier by me in [24]. The research direction b) was not considered in my study [24]. My newest paper [26] constituted supplement and generalization of the results, which were obtained earlier in [24] under lower assumptions. Komar [27] showed that special states are inevitable in the case of $\Lambda = 0$ under the absence of, in our terminology, the power field, absolute rotation, pressure, viscosity and the flow of energy. This conclusion repeats one of the results obtained earlier in [24] and [25].

In the next Paragraphs I give the further generalization and development of some results initially obtained by me in [24, 26].

This is Gödel's solution [22], which will be required in our research:

$$\left.\begin{array}{c} ds^2 = a^2\left[(dx^0)^2 + 2\,e^{x^1}dx^0dx^2 - (dx^1)^2 + \dfrac{e^{2x^1}}{2}\,(dx^2)^2 - (dx^3)^2\right] \\ 0 < a = const \end{array}\right\}. \qquad (3)$$

In cosmology, accompanying coordinates are commonly used. The necessary and sufficient condition for such coordinates requires that the numerical value of the three-dimensional velocity should be lower than the velocity of light, while the components of the velocity should be finite, simple and continuous functions of the four coordinates.

§3. We denote space-time indices 0, 1, 2, 3 in Greek (where 0 corresponds to the time dimension), while spatial indices 1, 2, 3 are denoted in Roman. We assume that summation takes a place on two same indices met in the same term. We assume that

$$x^0 = ct, \qquad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu,$$

and, in a locally Galilean reference frame, we have

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

We assume also that the components of the metric world-tensor $g_{\mu\nu}$ are continuous along the coordinates x^α in common with their first derivatives and second derivatives. In common, we assume that all quantities used here satisfy, in this part and in the rest parts, the requirements of the General Theory of Relativity.* Besides, while talking about three-dimensional (spatial) tensors and the other three-dimensional geometrical quantities (e.g. Christoffel's symbols), we will omit the notion about the number of the dimensions.

Four-dimensional coordinate systems resting with respect to the same reference body (which is deforming, in a general case) are connected to each other by the transformations

$$\tilde{x}^0 = \tilde{x}^0(x^0, x^1, x^2, x^3)\,, \tag{4a}$$

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, x^3)\,, \qquad \frac{\partial\,\tilde{x}^i}{\partial\,x^0} = 0\,. \tag{4b}$$

The choice of a body of reference is equivalent to the choice of the congruence of the time lines $x^i = const$. Suppose that a reference body has been chosen. Then, of all the quantities non-covariant to the general transformations

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^0, x^1, x^2, x^3)\,, \tag{5}$$

those quantities are physically preferred which are covariant with respect to the transformations (4a) and (4b). Hence, such physically preferred quantities are invariant with respect to the transformations (4a), and are covariant to the transformations (4b). We therefore call such physically preferred quantities *chronometric invariants*. Such chrono-

*The formulae (1), (2) and (3) satisfy all these requirements as well.

metrically invariant quantities can be considered as three-dimensional tensors in any of the given spatial sections $x^0 = const$. They can be also considered as tensors in a space, all elements of which (i.e. three-dimensional local spaces) are definitely orthogonal to the time lines under any given coordinate of time. We mean, by a three-dimensional space of a given body of reference (a reference space), a space determined in this way. Such a space is, generally speaking, non-holonomic. This means that, with a body of reference given in a general case, it is impossible to find such a spatial section which could be everywhere orthogonal to the time lines: in such a general case it is impossible to find, by the transformation (4a), such a coordinate of time x^0 that $g_{0i} = 0$ would be everywhere true in the spatial section.

With chronometrically invariant quantities and chronometrically invariant operators, we remove a difficulty proceeding from the fact that many non-chronometrically invariant quantities and relations (the conditions of homogeneity and isotropy, for instance) depend on the arbitrarity of our choice of the time coordinate. In a general case (in Gödel model, for instance), this difficulty can neither be avoided by the choice of a preferred coordinate of time satisfying the conditions $g_{00} = 1$ and $g_{0i} = 0$ (as for the homogeneous isotropic models) nor the substantial easing of this situation due to the choice of a preferred coordinate of time such that the weak condition $g_{0i} = 0$ satisfies everywhere.

Let $Q^{ik\dots p}_{00\dots 0}$ be the components of a world-tensor of the rank n, all upper indices of which are nonzero, while all m lower indices are zero. For such a tensor, the quantities

$$T^{ik\dots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\dots p}_{00\dots 0}$$

are the components of a chronometrically invariant contravariant (three-dimensional) tensor of the rank $n - m$. Using this rule, we can easily find the chronometrically invariant form for quantities and operators, if we know the formulae of them under a specially chosen coordinate of time according to the transformations (4a), for instance, if $g_{00} = 1$ and $g_{0i} = 0$ at the given world-point.

§4. Targeting the chronometrically invariant formulae for the elementary length $d\sigma$, the metric tensors h_{ik} and h^{ik}, and the fundamental determinant $h = |\,h_{ik}|$, we obtain

$$d\sigma^2 = h_{ik}\, dx^i dx^k, \tag{6}$$

$$h_{ik} = -\,g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}}\,, \quad h^{ik} = -\,g^{ik}, \quad h = -\frac{g}{g_{00}}\,, \tag{7}$$

where $g = |g_{\mu\nu}|$. The spatial metric determined in such a way coincides with that assumed by Landau and Lifshitz, see (82.5) and (82.6) in [28], and that assumed by Fock, see (55.20) in [29]. For the elementary chronometrically invariant interval of time $d\tau$ and the elementary world-interval ds, we obtain

$$c\,d\tau = \frac{g_{0\alpha}dx^{\alpha}}{\sqrt{g_{00}}}\,, \qquad ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{8}$$

For the chronometrically invariant velocity v^i of the motion of a test-particle, we have

$$\mathrm{v}^i = \frac{dx^i}{d\tau}\,, \qquad h_{ik}\,\mathrm{v}^i\mathrm{v}^k = \left(\frac{d\sigma}{d\tau}\right)^2.$$

If $ds = 0$, $h_{ik}\,\mathrm{v}^i\mathrm{v}^k = c^2$: the chronometrically invariant velocity of light in vacuum is always equal the fundamental velocity.

We mark the chronometrically invariant differential operators by the asterisk. For such operators (they coincide with d/dt, $\partial/\partial t$ and $\partial/\partial x^i$ under the conditions $g_{00} = 1$ and $g_{0i} = 0$) we obtain

$$\frac{^*d}{dt} = \frac{d}{d\tau}\,, \qquad \frac{^*\partial}{\partial t} = \frac{c}{\sqrt{g_{00}}}\frac{\partial}{\partial x^0}\,, \qquad \frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\frac{\partial}{\partial x^0}\,. \tag{9}$$

For the chronometrically invariant generalizations of Christoffel's symbols and the operator of general covariant differentiation, we have*

$$\Delta_{ij,k} = \frac{1}{2}\left(\frac{^*\partial h_{jk}}{\partial x^i} + \frac{^*\partial h_{ik}}{\partial x^j} - \frac{^*\partial h_{ij}}{\partial x^k}\right), \quad \Delta^k_{ij} = h^{kl}\Delta_{ij,l}\,, \tag{10}$$

$$^*\nabla_i\, Q^{\cdot\cdot\cdot k}_{j\dots} = \frac{^*\partial Q^{\cdot\cdot\cdot k}_{j\dots}}{\partial x^i} - \Delta^l_{ij}Q^{\cdot\cdot\cdot k}_{l\dots} - \dots + \Delta^k_{il}Q^{\cdot\cdot\cdot l}_{j\dots}\,. \tag{11}$$

As can be easily seen,

$$^*\nabla_i\, h_{jk} = 0\,, \qquad ^*\nabla_i\, h^k_j = 0\,, \qquad ^*\nabla_i\, h^{jk} = 0\,.$$

The metric of a spatial section $x^0 = const$ is determined by the tensor

$$y_{ik} = -\,g_{ik}\,, \quad y^{ik} = -\,g^{ik} + \frac{g^{0i}g^{0k}}{g^{00}}\,, \quad y = -\,g\,g^{00}, \tag{12}$$

where $y = |y^{ik}|$ is the determinant of the tensor.

The metric (7) is chronometrically invariant, space-like everywhere, and the length of an "unchangeable" elementary rest-scale in this metric equals the "proper" length. On the other hand, the metric (12) does not

*See also formula (17).

bear these properties: see, for instance, Gödel's model (3). In particular, in contrast to h, y may become negative (in general, $y \leqslant h$) that leads to the negative numerical value of the volume of the region of the spatial section where this happens. Therefore, even with a given coordinate of time, and fixation of the numerical value of the coordinate, a true physical meaning is attributed to not the metric (12) of a spatial section $x^0 = const$, but to the metric (7) of the space (space-time) where the value $x^0 = const$ is fixed.

§5. We assume that all differentiable quantities bear the properties which allow them to change the order in the usual (not chronometrically invariant or generally covariant) differentiation. In such a case,

$$\frac{^*\partial^2}{\partial x^i\,\partial t} - \frac{^*\partial^2}{\partial t\,\partial x^i} = \frac{F_i}{c^2}\frac{^*\partial}{\partial t}\,, \qquad \frac{^*\partial^2}{\partial x^i\,\partial x^k} - \frac{^*\partial^2}{\partial x^k\,\partial x^i} = \frac{2\,A_{ik}}{c^2}\frac{^*\partial}{\partial t}\,. \tag{13}$$

These chronometrically invariant vector F_i and chronometrically invariant antisymmetric tensor A_{ik}, determined by the equalities (9) and (13), satisfy the identities

$$\frac{^*\partial A_{jk}}{\partial x^i} + \frac{^*\partial A_{ki}}{\partial x^j} + \frac{^*\partial A_{ij}}{\partial x^k} + \frac{1}{c^2}\left(F_i A_{jk} + F_j A_{ki} + F_k A_{ij}\right) = 0\,, \tag{14}$$

$$\frac{^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{^*\partial F_k}{\partial x^i} - \frac{^*\partial F_i}{\partial x^k}\right), \tag{15}$$

and also to the identities (17).

The identity satisfying the three equalities $A_{ik} = 0$ in a given four-dimensional region is the necessary and sufficient condition for the reducing of all g_{0i} to zero everywhere in this region by the transformation (4a): in such a case $d\tau$ has an integration multiplier, i.e. time is allowed to be integrated along a path in this region (time is integrable). In other words, the identity satisfying the equalities $A_{ik} = 0$ is the necessary and sufficient condition of holonomity of the given space of reference. Thus A_{ik} is the chronometrically invariant tensor of the space non-holonomity. The identity satisfying all six equalities $F_i = 0$ and $A_{ik} = 0$ in a given four-dimensional region is the necessary and sufficient condition for the reducing of all g_{00} to 1 and of all g_{0i} to zero by the transformation (4a). In other words, this is necessary and sufficient for $d\tau$ to be a total differential.

At any world-point O, one can set up a four-dimensional locally geodesic frame of reference $\tilde{\Sigma}_0$, which satisfies the following condition: at this point, the chronometrically invariant velocity of a given reference frame Σ with respect to the locally geodesic reference frame $\tilde{\Sigma}_0$

is zero $(\tilde{\mathrm{v}}^j)_0 = 0$. Considering the reference frame $\tilde{\Sigma}_0$, we introduce in it the chronometrically invariant quantities which characterize the motion of our reference frame Σ with respect to $\tilde{\Sigma}_0$ in a four-dimensional neighbourhood of the point O: we take the generally covariant characteristics of the motion such as the acceleration vector $(\tilde{w}_j)_0$, the tensor of angular velocity of the rotation $(\tilde{a}_{jl})_0$ and the tensor of the rate of the deformation $(\tilde{d}_{jl})_0$, then express them through the chronometrically invariant velocity by the removing of regular derivatives with chronometrically invariant derivatives. Using the general transformations (5), we obtain that the equalities

$$F_i = -\frac{\partial \tilde{x}^j}{\partial x^i}\,(\tilde{w}_j)_0\,, \qquad A_{ik} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^k}\,(\tilde{a}_{jl})_0$$

are true at any world-point O.

We introduce also a chronometrically invariant tensor D_{ik}, which satisfies the equality

$$D_{ik} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^k}\,(\tilde{d}_{jl})_0$$

at any world-point O.

In this context, F_i is the vector of acceleration of our reference space Σ with respect to the locally geodesic reference space $\tilde{\Sigma}_0$, taken with the opposite sign, A_{ik} is the tensor of angular velocity of the rotation of our reference space Σ with respect to $\tilde{\Sigma}_0$, while D_{ik} is the tensor of the rate of deformation of our reference space Σ with respect to $\tilde{\Sigma}_0$. It is possible to prove that

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \tag{16}$$

where $D = D_j^j$ has the meaning of the speed of the relative expansion of the volume element of the space.*

Denote by $\Gamma^{\sigma}_{\mu\nu}$ the four-dimensional Christoffel symbols of the 2nd kind. Then we have the identities

$$\left.\begin{aligned} &\frac{\Gamma^i_{00}}{g_{00}} = -\frac{F^i}{c^2}\,, \qquad \frac{g^{i\alpha}\Gamma^k_{\alpha 0}}{\sqrt{g_{00}}} = -\frac{1}{c}\left(A^{ik} + D^{ik}\right) \\ &g^{i\alpha}g^{j\beta}\Gamma^k_{\alpha\beta} = h^{il}h^{jm}\Delta^k_{lm} \end{aligned}\right\}, \tag{17}$$

*The volume of an element of the space we are considering can be represented as an integral from $\sqrt{h}\,dx^1dx^2dx^3$, where dx^i and also the region of the change of x^1 along which the integration is processed are independent of x^0.

which allow us to find F_i, A_{ik}, D_{ik} and Δ^k_{lm} through $\Gamma^\sigma_{\mu\nu}$.

The study of the equations of motion of a particle, presented in [26], manifested that F^k can be interpreted as the sum of the force of gravity and the force of inertia (the latter is derived from the carrying acceleration), both calculated for the unit of mass, while A_{ik} is the angular velocity of the absolute rotation of our reference frame derived from Coriolis' effect.

§6. For a covariant vector Q_l, with the note on the properties of the differentiable quantities we made in the beginning of §5, we obtain

$$\left({}^*\nabla_{ik} - {}^*\nabla_{ki}\right) Q_l = \frac{2\,A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdot\cdot\cdot j}\,Q_j\;, \tag{18}$$

$$H_{lki\cdot}^{\cdot\cdot\cdot j} = \frac{{}^*\partial \Delta^j_{il}}{\partial x^k} - \frac{{}^*\partial \Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}\;. \tag{19}$$

The chronometrically invariant tensor (19), which is analogous to Schouten's tensor, is different in its properties in a general case from the Riemann-Christoffel tensor. We introduce the chronometrically invariant tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \tag{20}$$

which possesses all the algebraical properties of the Riemann-Christoffel tensor. There are identity correlations between the quantities H_{lkij}, from one side, and also the quantities C_{lkij}, D_{mn} and A_{mn} from the other side. As easy to see,

$$H_{lkij} = C_{lkij} + \frac{1}{c^2}\left(2A_{ki}D_{jl} + A_{ij}D_{kl} + A_{jk}D_{il} + A_{kl}D_{ij} + A_{li}D_{jk}\right). \tag{21}$$

As obvious, if $A_{mn}=0$ or $D_{mn}=0$, we have $H_{lkij}=C_{lkij}$. We introduce also $H_{lk}=H_{lki\cdot}^{\cdot\cdot\cdot i}$, $H=H^k_k$ and $C_{lk}=C_{lki\cdot}^{\cdot\cdot\cdot i}$, $C=C^k_k$. Then

$$H_{lk} = C_{lk} + \frac{1}{c^2}\left(A_{kj}D^j_l + A_{lj}D^j_k + A_{kl}D\right), \quad H = C\,. \tag{22}$$

The metric of any spatial section is determined by (12). The curvature of a spatial section is characterized by the regular Riemann-Christoffel tensor K_{lkij} corresponding to the metric (12). Pave such spatial sections $x^0 = const$ through a world-point O, but at different coordinates of time, that they satisfy (at this point O) the conditions

$$g_{0i} = 0\,, \qquad \frac{\partial g_{0i}}{\partial x^k} + \frac{\partial g_{0k}}{\partial x^i} = 0\,. \tag{23}$$

We call such spatial sections maximally orthogonal to the time line in a neighbourhood of the given world-point. Each of the spatial sections possesses its own Riemann-Christoffel tensor K_{lkij}. These tensors coincide with each other at this point O, and satisfy the equality

$$C_{lkij} = K_{lkij} + \frac{2}{c^2}\left(A_{ij}A_{kl} + A_{jk}A_{il} + 2A_{ik}A_{jl}\right). \tag{24}$$

In each of these maximally orthogonal spatial sections, which cross the space at the point O, we introduce the regular Riemann-Christoffel tensor corresponding to the metric (7), not to (12).* This tensor can be considered as the Riemann-Christoffel tensor of a space, wherein the coordinate of time is fixed at a numerical value $x^0 = const$, satisfying the conditions (23). At the world-point O, the tensors coincide with each other in all the spatial sections, and are equal to C_{lkij}. Let x^{mn} be a chronometrically invariant unit bivector, which fixes a two-dimensional direction in a given spatial section. In such a case, for the Riemannian curvature in this two-dimensional direction, we have $K_{lkij}\, x^{ik}x^{lj}$ in the metric (12) and $C_{lkij}\, x^{ik}x^{lj}$ in the metric (7). Due to (21) and (24),

$$H_{lkij}\, x^{ik}x^{lj} = C_{lkij}\, x^{ik}x^{lj} = K_{lkij}\, x^{ik}x^{lj} - \frac{12}{c^2}\left(A_{ij}x^{ij}\right)^2. \tag{25}$$

We introduce also $K_{lk} = K_{lki\cdot}^{\cdots i}$ and $K = K_k^k$. In such a case,

$$C_{lk} = K_{lk} + \frac{6}{c^2}\, A_{li}A_{k\cdot}^{\cdot i}\,, \qquad C = K + \frac{6}{c^2}\, A_{ki}A^{ki}. \tag{26}$$

For the Gaussian curvatures, we have, respectively: $-\frac{1}{6}\,C$ and $-\frac{1}{6}\,K$. As obvious,

$$C_{lkij}\, x^{ik}x^{lj} \leqslant K_{lkij}\, x^{ik}x^{lj}, \qquad -\frac{1}{6}\,C \leqslant -\frac{1}{6}\,K\,.$$

Thus, with a fixed A_{mn}, the space curvature is characterized by the quantities C_{lkij}, C_{lk} and C, which are connected to the metric (7), and also by the quantities K_{lkij}, K_{lk} and K, connected to the metric (12). Because the metric (7) is physically preferred, we will use those quantities, which are connected to it.

§7. Here we introduce auxiliary quantities and relations.

Let ε_{ijk} and ε^{ijk} be such antisymmetric unit chronometrically invariant tensors that $\varepsilon_{123} = \sqrt{h}$ and $\varepsilon^{123} = 1/\sqrt{h}$. As easy to see,

$$^*\nabla_l\, \varepsilon_{ijk} = 0\,, \quad ^*\nabla_l\, \varepsilon^{ijk} = 0\,, \quad \frac{^*\partial \varepsilon_{ijk}}{\partial t} = \varepsilon_{ijk}\, D\,, \quad \frac{^*\partial \varepsilon^{ijk}}{\partial t} = -\,\varepsilon^{ijk} D\,.$$

*Generally speaking, the metrics (7) and (12) coincide only at the point O.

We introduce the chronometrically invariant vector of angular velocity of the space rotation

$$\Omega^i = \frac{1}{2}\,\varepsilon^{ijk} A_{jk}\,, \qquad \Omega_i = \frac{1}{2}\,\varepsilon_{ijk} A^{jk}. \tag{27}$$

The identities (14) and (15) can be represented, respectively, as

$$^*\nabla_j\,\Omega^j + \frac{1}{c^2}\,F_j\,\Omega^j = 0\,, \tag{28}$$

$$\frac{2}{\sqrt{h}}\,\frac{^*\partial}{\partial t}\left(\sqrt{h}\,\Omega^i\right) + \varepsilon^{ijk}\,{}^*\nabla_j\,F_k = 0\,. \tag{29}$$

As obvious, any axial vector field is anisotropic. The field of a tensor Z_{ik} is isotropic, if $Z_{ik} = \frac{1}{3}\,Z h_{ik}$, where $Z = Z_j^j$. We characterize the anisotropy of the space deformation and the anisotropy of the space curvature by the quantities, respectively,

$$\Pi_{ik} = D_{ik} - \frac{1}{3}\,Dh_{ik}\,, \quad \Pi_i^k \Pi_k^i = D_i^k D_k^i - \frac{1}{3}\,D^2 \geqslant 0\,, \tag{30}$$

$$\Sigma_{ik} = C_{ik} - \frac{1}{3}\,C\,h_{ik}\,. \tag{31}$$

The condition of homogeneity of the field of any tensor $Z_{i\dots}^{\cdot\cdot k}$ can be written in the form: $^*\nabla_j\,Z_{i\dots}^{\cdot\cdot k} = 0$.

We assume the notations

$$\dot{Z} = \frac{\partial Z}{\partial t}\,, \qquad {}^*\dot{Z} = \frac{^*\partial Z}{\partial t}\,. \tag{32}$$

As obvious, having any chosen coordinate of time, the conditions $^*\dot{Z} = 0$ and $^*\ddot{Z} > 0$ are equivalent to the conditions $\dot{Z} = 0$ and $\ddot{Z} > 0$, the conditions $^*\dot{Z} = 0$ and $^*\ddot{Z} = 0$ are equivalent to the conditions $\dot{Z} = 0$ and $\ddot{Z} = 0$, while the conditions $^*\dot{Z} = 0$ and $^*\ddot{Z} < 0$ are equivalent to the conditions $\dot{Z} = 0$ and $\ddot{Z} < 0$. Thus, marking the time derivatives by the asterisk, we can write the conditions of the extrema in the chronometrically invariant form.

§8. We denote by ρ the density of mass, by J^i the vector of the density of the flow of mass (this quantity is the same that the vector of the density of momentum), by U^{ik} the tensor of the density of the flow of momentum, and $U = U_j^j$. As obvious, ρc^2 is the density of energy, while $J^i c^2$ is the vector of the density of the flow of energy. Let these notations be attributed to chronometrically invariant quantities.

In such a case,

$$\frac{T_{00}}{g_{00}} = \rho\,, \qquad \frac{c\,T_0^i}{\sqrt{g_{00}}} = J^i, \quad c^2 T^{ik} = U^{ik}, \quad T = \rho - \frac{U}{c^2}\,.$$

The equations of the conservation of energy and momentum can be written as follows

$$\frac{^*\partial\rho}{\partial t} + D\rho + \frac{1}{c^2}\,D_{ij}U^{ij} + \left[\left(^*\nabla_j - \frac{1}{c^2}\,F_j\right) J^j\right] - \frac{1}{c^2}\,F_j J^j = 0\,, \tag{33}$$

$$\frac{^*\partial J^k}{\partial t} + DJ^k + 2\big(D_i^k + A_{i\cdot}^{\cdot k}\big)J^i + \left[\left(^*\nabla_i - \frac{1}{c^2}\,F_i\right) U^{ik}\right] - \rho F^k = 0\,. \tag{34}$$

All the terms contained on the left side of the equations (33) and (34) have obvious physical meanings. The third term and the fifth (last) term in (33) are the relativistic terms, proceeding from the connexion between mass and energy. These terms take into account the change of the density of mass, which is due to the surface forces working while the volume element of space deforms (the third term on the left side), and the change of the flowing energy due to the acting gravitational and inertial forces (the fifth term). The fourth terms of (33) and (34) (they are taken into square brackets) constitute the "physical divergence" of J^i and U^{ik} respectively. The fact that physical divergence differs from mathematical divergence originates in the circumstance that, with the same dt, the intervals $d\tau$ are different at different coordinate points on the boundary of the elementary volume. As is obvious, (33) and (34) are the actual equations for mass and momentum. Multiplying (33) and (34) term-by-term by c^2, we are able to obtain the actual equations for energy and the flow of energy.

With all the above, Einstein's equations (1) take the form

$$\frac{^*\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + {}^*\nabla_j\,F^j - \frac{1}{c^2}\,F_jF^j = \\ = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \Lambda c^2, \tag{35}$$

$$^*\nabla_j\big(h^{ij}D - D^{ij} - A^{ij}\big) + \frac{2}{c^2}\,F_jA^{ij} = \varkappa J^i, \tag{36}$$

$$\frac{^*\partial D_{ik}}{\partial t} - \big(D_{ij} + A_{ij}\big)\big(D_k^j + A_{k\cdot}^{\cdot j}\big) + DD_{ik} - D_{ij}D_k^j + \\ + 3A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left(^*\nabla_i\,F_k + {}^*\nabla_k\,F_i\right) - \frac{1}{c^2}\,F_iF_k - c^2C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right) + \Lambda c^2 h_{ik}\,. \tag{37}$$

As obvious, all these ten equations (35), (36) and (37) are connected by four relations (33) and (34).

In a coordinate frame accompanying a medium, this medium plays a rôle of the body of reference, while the world-lines of the elements of this medium are the lines of time. In such a reference frame, the tensors D_{ik} and A_{ik} characterize the rate of deformation and the velocity of rotation of the medium. The equations (35), (36) and (37) in such accompanying coordinates can be considered as the equations of motion of a continuous medium. These equations, in common with the equations (33) and (34) and also the identities (14) and (15), allow a far-reaching analogy with the non-relativistic equations of motion of a continuous medium which satisfy the unitary interpretation [30]. This analogy permits the non-relativistic equations to be used for the quasi-Newtonian approximation to the relativistic theory (see §18).

§9. So forth, we use a coordinate frame, which accompanies a medium. We assume also that there are no other forces produced by masses, aside for the gravitational and inertial forces. We characterize the following chronometrically invariant quantities: the equilibrium pressure p_0 (it is determined by the equation of state), the true pressure p, the tensor of the viscous tension α_{ik}, the anisotropic part $\beta_{ik} = \alpha_{ik} - \frac{1}{3}\alpha h_{ik}$ of the viscosity tensor α_{ik}, and also $\alpha = \alpha_j^j$. With these notations,

$$p = p_0 - \frac{1}{3}\,\alpha\,, \quad U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}\,.$$

Using (33) and thermodynamical considerations, we obtain

$$\beta_{jl} D^{jl} = \beta_{jl}\Pi^{jl} \geqslant 0\,, \qquad \alpha D \geqslant 0\,. \tag{38}$$

In the coordinates, accompanying the medium, $c^2 J^i = q^i$, where q^i is the vector of density of the flow of any form of energy (radiation or heat) with respect to this medium. The viscosity, characterized by the tensor β_{ik}, and the viscosity, characterized by the scalar α, can be considered as the viscosity of the 1st kind and that of the 2nd kind, respectively. Thus the conservation equations (33) and (34) take the form

$$\frac{{}^*\partial\rho}{\partial t} + D\left(\rho + \frac{p}{c^2}\right) = \frac{1}{c^2}\left[\beta_{jl}\,\Pi^{jl} - \left({}^*\nabla_j\, q^j - \frac{2}{c^2}\,F_j\, q^j\right)\right], \tag{39}$$

$$\frac{1}{c^2}\left(\frac{{}^*\partial q_i}{\partial t} + D q_i - 2A_{i\cdot}^{\cdot j} q_j\right) - \left({}^*\nabla_j - \frac{1}{c^2}\,F_j\right)\beta_i^j + {} \\ + \left(\frac{{}^*\partial p}{\partial x^i} - \frac{1}{c^2}\,F_i\, p\right) - \rho F_i = 0\,, \tag{40}$$

while the system of Einstein's equations (35), (36) and (37) is equivalent to the system

$$\frac{^*\partial D}{\partial t} + \frac{1}{3}D^2 + \Pi_{jl}\Pi^{jl} - A_{jl}A^{jl} + {}^*\nabla_j F^j - \frac{1}{c^2}F_jF^j = -\frac{\varkappa}{2}\left(\rho c^2 + 3p\right) + \Lambda c^2, \quad (41)$$

$$\frac{4}{3}\frac{^*\partial D}{\partial x^i} - {}^*\nabla_j\left(\Pi_i^j + A_{i\cdot}^{\cdot j}\right) + \frac{2}{c^2}F_jA_{i\cdot}^{\cdot j} = \frac{\varkappa}{c^2}q_i\,, \quad (42)$$

$$\frac{1}{3}D^3 - \frac{1}{2}\Pi_{jl}\Pi^{jl} + \frac{3}{2}A_{jl}A^{jl} - \frac{1}{2}c^2C = \left(\varkappa\rho + \Lambda\right)c^2, \quad (43)$$

$$\frac{^*\partial \Pi_i^k}{\partial t} + D\Pi_i^k + \Pi_{ij}A^{kj} + \Pi^{kj}A_{ij} + 2\left(A_{ij}A^{kj} - \frac{1}{3}A_{jl}A^{jl}h_i^k\right) + \left[\frac{1}{2}\left({}^*\nabla_i F^k + {}^*\nabla^k F_i\right) - \frac{1}{3}\left({}^*\nabla_j F^j\right)h_i^k\right] - \frac{1}{c^2}\left(F_iF^k - \frac{1}{3}F_jF^jh_i^k\right) - c^2\Sigma_i^k + \varkappa\beta_i^k = 0\,. \quad (44)$$

The equations (41) and (42) are the actual equations (35) and (36) transformed with (30). The equations (43) and (44) were obtained from (35) and (37) with the use of (30) and (31). The left side of (44) is a tensor, whose trace is identically equal to zero. Thus all six equations, which constitute (44), are connected by the same algebraic relation.

§10. The equations (41) and (44) set up a connexion between the fields F_i, A_{ik}, D_{ik} and C_{ik}, from the one side, and the fields ρ, p, β_{ik} and q_i, from the other side. It is natural to determine the homogeneity of the Universe, in a given local region of it, by the conditions

$$\left.\begin{array}{llll} {}^*\nabla_j F_i = 0\,, & {}^*\nabla_j A_{ik} = 0\,, & {}^*\nabla_j D_{ik} = 0\,, & {}^*\nabla_j C_{ik} = 0 \\ \dfrac{^*\partial\rho}{\partial x^i} = 0\,, & \dfrac{^*\partial p}{\partial x^i} = 0\,, & {}^*\nabla_j \beta_{ik} = 0\,, & {}^*\nabla_j q_i = 0 \end{array}\right\}, \quad (45)$$

while the isotropy of the Universe, in a given local region, can be determined by the conditions

$$F_i = 0\,, \quad A_{ik} = 0\,, \quad \Pi_{ik} = 0\,, \quad \Sigma_{ik} = 0\,, \quad \beta_{ik} = 0\,, \quad q_i = 0\,. \quad (46)$$

As obvious, if we remove C_{ik} with K_{ik} in (45), the new conditions of the homogeneity will be equivalent to the initial conditions. Thus, if we remove the fourth condition of (46) with the requirement

$$K_{ik} - \frac{1}{3}Kh_{ik} = 0\,,$$

the new conditions of the isotropy will be equivalent to the initial conditions. As easy to see, there are six factors of the anisotropy: the power field, the absolute rotation, the anisotropy of the deformation, the anisotropy of the curvature, the viscosity and the 1st kind, and the flow of the energy. The first five factors are connected among themselves by the relations (44).

Let the conditions (46) be true everywhere in a finite or infinite four-dimensional region. In such a case, in the same region, the equations (44) are satisfied identically, while the equations (39–43) take the form

$$\frac{{}^*\partial\rho}{\partial t} + D\left(\rho + \frac{p}{c^2}\right), \tag{47}$$

$$\frac{{}^*\partial p}{\partial x^i} = 0\,, \tag{48}$$

$$\frac{{}^*\partial D}{\partial t} + \frac{1}{3}D^2 = -\frac{\varkappa}{2}\left(\rho c^2 + 3p\right) + \Lambda c^2, \tag{49}$$

$$\frac{{}^*\partial D}{\partial x^i} = 0\,, \tag{50}$$

$$\frac{1}{3}D^2 - \frac{1}{2}c^2 C = \left(\varkappa\rho + \Lambda\right)c^2. \tag{51}$$

It follows, from (48), (49) and (50), that

$$\frac{{}^*\partial\rho}{\partial x^i} = 0\,, \qquad \frac{{}^*\partial C}{\partial x^i} = 0\,, \tag{52}$$

where the last equality can be obtained also in a direct way, on the basis of Schur's theorem, due to the holonomity of this space, and the isotropy of its curvature.

The equations (48), (50), (52) and (46) lead immediately to the conditions of the homogeneity (45). On the basis of (47) and also (49), (50) and (51), while taking the third equality of (16) into account, we obtain

$$\frac{{}^*\partial\left(C\sqrt[3]{h}\right)}{\partial t} = 0\,. \tag{53}$$

If the condition (53), the first four conditions of the isotropy (46), and the second condition of (52) satisfy, there among the accompanying coordinate frames is such a frame, wherein the homogeneous isotropic metric (2) is true and also

$$D = 3\,\frac{\dot{R}}{R}\,, \qquad C = -\frac{6\,k}{R^2}\,. \tag{54}$$

For Gödel's model (3), we obtain

$$\left.\begin{array}{lllll} h_{11} = a^2, & h_{22} = \dfrac{a^2}{2}\,e^{2x^1}, & h_{33} = a^2, & h_{ik} = 0 & (i \neq k) \\ F_i = 0\,, & A_{12} = -\dfrac{ac}{2}\,e^{x^1}, & A_{23} = 0\,, & A_{31} = 0\,, & D_{ik} = 0 \\ C_{11} = 1\,, & C_{22} = \dfrac{1}{2}\,e^{2x^1}, & C_{33} = 0\,, & C_{ik} = 0 & (i \neq k) \\ \varkappa\rho = \dfrac{1}{a^2} = -2\Lambda\,, & & p = 0\,, & \beta_{ik} = 0\,, & q_i = 0 \end{array}\right\}. \tag{55}$$

As can be seen, here the second and fourth conditions of the conditions of the anisotropy (46) do not satisfy, while all the conditions of the homogeneity (45) are satisfied.

So, in a general case, we formulate the following conclusions about a four-dimension region of space: 1) the isotropy leads to the homogeneity, hence 2) the inhomogeneity leads to the anisotropy; 3) the anisotropy does not require inhomogeneity; 4) as aforementioned in this Paragraph, in the understanding of the homogeneity and isotropy, only the models (2) are homogeneous and isotropic, while the model (3) is homogeneous, but anisotropic.

§11. The vectorial equation of conservation (40) expresses the law of the change of the flow of energy with time. In the absence of such a flow, this equation expresses the equilibrium condition between the surface forces and the gravitational inertial force (it is originated in masses). The chronometrically invariant rotor of the vector of the gravitational inertial force, i.e. the tensor

$$^*\nabla_i\,F_k - {}^*\nabla_k\,F_i = \frac{^*\partial F_k}{\partial x^i} - \frac{^*\partial F_i}{\partial x^k}$$

or the vector $\varepsilon^{ijk}\,{}^*\nabla_j\,F_k$ is nonzero in a general case.

A local centre of gravitational attraction can be determined by the conditions ${}^*\nabla_j\,F^j < 0$ and $F_i = 0$. As obvious, at such a point and also in a neighbourhood surrounding it, the following condition is true

$$^*\nabla_j\,F^j - \frac{1}{c^2}\,F_j F^j < 0\,. \tag{56}$$

A local centre of radiation can be determined by the conditions ${}^*\nabla_j\, q^j < 0$ and $q_i = 0$. Hence, at such a point and also in a neighbourhood surrounding it, the following condition is true

$$ {}^*\nabla_j\, q^j - \frac{2}{c^2}\, F_j\, q^j > 0\,. \tag{57} $$

The scalar equation of conservation (39) expresses the law of the change of the mass or, equivalently, the energy of the volume element of the medium with time. We introduce the volume V and the energy $E = V\rho c^2$ of such an element. Taking into account that $D = {}^*\partial \ln V/\partial t$, we reduce (39) to the form

$$ dE + p\,dV = \left[\beta_{jl}\,\Pi^{jl} - \left({}^*\nabla_j\, q^j - \frac{2}{c^2}\, F_j\, q^j\right)\right] V d\tau\,, \tag{58} $$

where $p\,dV = p_0\, dV - \alpha D V d\tau$. As the inequalities (38) and (57) satisfy, and the sign of $d\tau$ is definitely given, the right side of (58) may reach, generally speaking, any sign. At the moment of an extremum of the volume of the element, obviously $D = 0$. At the moment of an extremum of the density of the volume, ${}^*\partial\rho/\partial t = 0$. As easy to see, from the scalar equation of conservation (39), these moments of time do not coincide in a general case.

In the absence of the viscosity of the 1st kind and also the flow of the energy, the equations (40) and (58) take the form, respectively,

$$ \frac{{}^*\partial p}{\partial x^i} = \left(\rho + \frac{p}{c^2}\right) F_i\,, \qquad dE + p\,dV = 0\,. \tag{59} $$

As well-known, the second of these equations was obtained earlier in the case of the metric (2), i.e. in the framework of the theory of a homogeneous isotropic universe.

§12. Consider the identities (14) and (15), and also the identities (28) and (29) which are equivalent to the previous. We see in (28) that, in a general case, neither the mathematical chronometrically invariant divergence nor the physical chronometrically invariant divergence of the vector of angular velocity of the absolute rotation of the space are non-equal to zero. In a particular case, in the absence of the power field, both divergences coincide, and are equal to zero. The identities (15) and (29), in the framework of the accompanying coordinates, represent the equations of the change of a vortex. In the case of a non-viscous barotropic medium free of the flow of energy, these identities give

$$ \frac{{}^*\partial}{\partial t}\left[A_{jk}\,(E + pV)\right] = 0\,, \qquad \frac{{}^*\partial}{\partial t}\left[\Omega^i\sqrt{h}\,(E + pV)\right] = 0\,. \tag{60} $$

These equalities are equivalent to each other. The second of them manifests the conservation of the vortical lines. The stress of the vortical tube is expressed by a surface integral from the quantity

$$\varepsilon_{ijk}\,\Omega^i dx^j \delta x^k = \frac{1}{2}\,A_{jk}\big(dx^j \delta x^k - dx^k \delta x^j\big)\,,$$

where the components of the vectors dx^k and δx^j are independent of time. Having the vortical lines conserved, the region of the change of the space coordinates, with respect to which we perform integration, does not depend on time. Therefore each of two tensor equalities (60) is the necessary and sufficient condition for the synchronous conservation of a) the vortical lines and b) the product of the multiplication of the stress of the vortical tubes by the relativistic heat function $E + pV$. In the absence of the power field, the identities (15) and (29) give

$$\frac{{}^*\partial A_{ik}}{\partial t} = 0\,, \qquad \frac{{}^*\partial}{\partial t}\big(\Omega^i \sqrt{h}\big) = 0\,. \tag{61}$$

In both cases (60) and (61), the conditions of the holonomity or the non-holonomity of the accompanying space remain unchanged: these conditions are free to be realized in both cases. If we suppose that the space is holonomic and the holonomity remains unchanged, this supposition leads to the other limitations, most artificial of which are the requirements for the non-viscous and barotropic properties of the medium in the absence of the flow of energy. These requirements satisfy, with high precision, the observed values of the density, the pressure and the parameters of expansion of that part of the Metagalaxy, which is accessible to our observation in the present epoch. On the other hand, these requirements satisfy the worse; the more earlier stage of the expansion is under our consideration. This is because, with the expansion of the Metagalaxy, the pressure decreases faster than the density. Hence, considering the ancient age of the Metagalaxy, we should mean the accompanying space of the Metagalaxy to be non-holonomic, that is equivalent to the supposition that the Metagalaxy rotates.

§13. Instead of the change of the volume V of an element of the medium, we will consider the change of the quantity $R = f\sqrt[3]{V}$, where $f > 0$ and $\partial f/\partial t = 0$. In such a case, $D = 3\,{}^*\dot{R}/R$. It is obvious that this quantity R, in contrast to the same named quantity of the formulae (2) and (54), considered under the condition $k \neq 0$, is determined at every point of the space to within a constant positive multiplier.

Our task is a general bound of the evolution of some characteristics of the space in the process of the expansion of the medium. We therefore consider this problem under some simplifications. We consider evolution of the factors of the anisotropy in the case where the rest factors of the anisotropy are neglected. Preliminary, we consider the change of the curvature scalar and the density under simplest assumptions.

In the case where the space is completely isotropic, as can be seen from (53) and (54), we have

$$C \sim R^{-2}. \tag{62}$$

If $p=0$, $\beta_{ik}=0$ and $q_i=0$, as seen from (39), we have

$$\rho \sim R^{-3}. \tag{63}$$

If $F_i=0$ and $\Pi_{ik}=0$, with use of (61) we obtain

$$\Omega_j\,\Omega^j \sim R^{-4}. \tag{64}$$

If $F_i=0$, $A_{ik}=0$, $\Sigma_{ik}=0$ and $\beta_{ik}=0$, it follows from (44) that

$$\Pi_i^k\,\Pi_k^i \sim R^{-6}. \tag{65}$$

As easy to see from (44), in the case where $F_i=0$, $A_{ik}=0$, $\Sigma_{ik}=0$ and $\beta_{ik}\neq 0$, the quantity $\Pi_i^k\Pi_k^i$ changes faster with the increasing R and slower with the decreasing R than according to the law (65). At $\beta_{ik}=2\eta\Pi_{ik}$, where η is the viscosity coefficient of the 1st kind, the quantity $\beta_i^k\beta_k^i$ changes faster than $\Pi_i^k\Pi_k^i$. If $F_i=0$, $p=0$ and $\beta_{ik}=0$, we obtain from (40) that

$$q_j\,q^j \sim R^{-8}. \tag{66}$$

Thus, according to our bound, the expansion of the Metagalaxy should be accompanied by a so fast decrease of the factors of the anisotropy such that the fact of the invisibility of the factors in the modern epoch does not allow us to ignore the presence of the factors in the past.

§14. We introduce the quantities

$$Q = \frac{2}{3}\,R\left(\Pi_i^k\,\Pi_k^i - 2\,\Omega_j\,\Omega^j + {}^*\nabla_j\,F^j - \frac{1}{c^2}\,F_jF^j\right), \tag{67}$$

$$S = \frac{1}{3}\,R^2\left(3\,\Omega_j\,\Omega^j - \frac{1}{2}\,\Pi_i^k\,\Pi_k^i - \frac{c^2}{2}\,C\right). \tag{68}$$

With these, the equations (39), (41) and (43) can be represented in the form, respectively,

$${}^*\dot{\rho} + 3\,\frac{{}^*\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = \frac{1}{c^2}\left[\beta_{jl}\,\Pi^{jl} - \left({}^*\nabla_j\,q^j - \frac{2}{c^2}\,F_j\,q^j\right)\right], \tag{69}$$

$$3\,\frac{{}^*\ddot{R}}{c^2R}+\frac{3}{2}\,\frac{Q}{c^2R}=-\frac{\varkappa}{2}\left(\rho+3\,\frac{p}{c^2}\right)+\Lambda\,, \tag{70}$$

$$3\,\frac{{}^*\dot{R}^2}{c^2R^2}+3\,\frac{S}{c^2R^2}=\varkappa\rho+\Lambda\,. \tag{71}$$

The equation (69), which connects the equations (70) and (71), can be transformed into the form

$${}^*\dot{S}={}^*\dot{R}Q+\frac{\varkappa}{3}R^2\left[\beta_{jl}\,\Pi^{jl}-\left({}^*\nabla_j\,q^j-\frac{2}{c^2}\,F_j\,q^j\right)\right]. \tag{72}$$

As obvious, having any initially chosen moment of time, we can set up such an initial value of R such that the initial value of S is equal to $kc^2=0,\pm c^2$. In the case where the space is completely isotropic, the right side of (69) and the second term on the left side of (70) are zero, while S still retains its initial numerical value. In such a case, we, omitting the asterisk, obtain the well-known equations for the homogeneous isotropic models (2). With the metric (2) the equations (40), (42) and (44) become identities. In such a case, two equations (70) and (71) of the whole scope of ten equations are sufficient, under some additional physical assumptions, for the investigation about the possible evolution of R with time. In a general case, we can also find the kinds of evolution of R permitted by the equations (70) and (71). We however should take into account the fact that, in the consideration of the whole system of ten equations of gravitation, we can find some of the kinds of the evolution to be impossible. Following in this way, we, obviously, narrow the circle of the conceivable possibilities step-by-step.

From cosmological and cosmogonical points of view, most interesting are the principal possibility and the physical conditions in a) the models of the kind O_2 that points to an oscillation between two regular extrema of R at finite numerical values of the density (the so-called "oscillation of the 2nd kind"), or at least the principal possibility and the physical conditions of b) a regular minimum of R at a finite numerical value of the density. There is also an important question about the principal possibility and the physical conditions of c) the accelerating increase of R, because such a growth at the current velocity of the expansion of the Metagalaxy leads to the prolongation of the past part of the epoch of the expansion, i.e. to the prolongation of the whole scale of time. As known, for the homogeneous isotropic models (2) considered in the framework of the suppositions

$$0<\rho c^2\geqslant 3p\geqslant 0\,,\qquad \frac{\partial p}{\partial R}\leqslant 0\,, \tag{73}$$

the case a) is impossible, while the cases b) and c) are permitted with only a positive numerical value of the cosmological constant.

Let the cosmological constant be zero. In such a case, on the basis of (67–71), we obtain that in the absence of the absolute rotation and the inequality (56)* the cases b) and c) are impossible and, hence, the case a) is impossible as well. The numerical value of R increases either monotonically and, at $\tau \to \infty$, becomes unbounded, or it increases till a regular maximum, and then decreases. If in addition to it, the medium is free of viscosity and the flow of energy, the beginning of the increase and the end of the decrease of R is so-called a "special state", where the density and the speed of the change of R are infinite. In such a case, we obtain the same two kinds of evolution as those known in the models (2): M_1 that means the "monotonic change of the 1st kind", and also O_1, i.e. the "oscillation of the 1st kind". In this process, the anisotropy of the deformation leads to more braking of the expansion and, hence, to the shortening the whole scale of time. Thus, in concern of the accelerating expansion, the regular minima and the oscillation of the 2nd kind, the most important is the taking of the power field and the absolute rotation into account. Concerning the irregular minima free of the special states, most important is the taking of the viscosity and the flow of energy into account.

§15. In this Paragraph we consider the kinds of evolution of R in complete as permitted by the equations (70) and (71) in the framework of the suppositions (73). We consider the case of a barotropic non-viscous medium, which is free of the flows of energy. In such a case, the density and the pressure at each point can be considered as functions of R. From (69), we obtain

$$\frac{\partial\rho}{\partial R} + \frac{3}{R}\left(\rho + \frac{p}{c^2}\right) = 0\,, \qquad \frac{\partial}{\partial R}\left(\rho R^3\right) = -\,3R^2\,\frac{p}{c^2}\,. \tag{74}$$

In the absence of the pressure, the density changes according to (69), i.e. this process goes faster under the positive pressure. It is obvious that, if $R \to \infty$, $\rho R^n \to 0$ and $pR^n \to 0$ (here $0 \leqslant n \leqslant 3$). We define R_∞ as $R \to R_\infty$ under $\rho \to \infty$. With this definition we see that $R_\infty \geqslant 0$. As obvious, at the value $R \to R_\infty$ we have $\rho R^n \to \infty$ $(0 \leqslant n \leqslant 3)$.

Given the plane RS, we consider the area of the real changes of the volume of the space element, i.e. such an area wherein $R \geqslant R_\infty$ and ${}^*\dot{R}^2 \geqslant 0$. This area is bounded by the ultimate lines: the straight line

*This can be if, for instance, in addition to the absence of the absolute rotation, there is no the power field.

$R=R_\infty$ and the ultimate curve ${}^*\dot{R}^2=0$. Denoting by S_0 the ordinate of a point on this ultimate curve ${}^*\dot{R}^2=0$, we express the equation of this curve through (71) as follows

$$S_0 = \frac{c^2}{3}\left(\varkappa\rho + \Lambda\right)R^2, \qquad R \geqslant 0\,. \tag{75}$$

While taking (74) into account, we obtain

$$\frac{\partial S_0}{\partial R} = \frac{c^2}{3}\left[-\varkappa\left(\rho + 3\,\frac{p}{c^2}\right) + 2\Lambda\right]R\,, \tag{76}$$

$$\frac{\partial^2 S_0}{\partial R^2} = \frac{c^2}{3}\left[\varkappa\left(2\rho - \frac{3}{c^2}\frac{\partial p}{\partial R}R\right) + 2\Lambda\right] = 2\,\frac{S_0}{R^2} - \varkappa R\,\frac{\partial p}{\partial R}\,, \tag{77}$$

where, due to the second of the suppositions (73),

$$2\rho - \frac{3}{c^2}\frac{\partial p}{\partial R}R > 0\,. \tag{78}$$

At any value of Λ we obtain $S_0 \to +\infty$, while $\partial S_0/\partial R \to -\infty$ at $R \to R_\infty$: the ultimate straight line is the asymptote of the ultimate curve. If $\Lambda > 0$, with the increasing of R to the value R_∞ the value of S_0 decreases up to its minimal value $\frac{\varkappa}{2}\left(\rho c^2 + p\right)R^2$ then monotonically increases: $S_0 \to +\infty$ and $\partial S_0/\partial R \to +\infty$ at the value $R \to \infty$. In such a case the ultimate curve lies above the axis of abscisses, and is convex everywhere to the axis. If $\Lambda = 0$, with the increasing of R to the value R_∞ the value of S_0 monotonically decreases: $S_0 \to 0$ and $\partial S_0/\partial R \to 0$ at the value $R \to \infty$. In such a case, the axis of abscisses is the asymptote: the ultimate curve lies above this axis, and is convex everywhere to it. If $\Lambda < 0$, with the increasing of R to the value R_∞ the value of S_0 monotonically decreases: $S_0 \to -\infty$ and $\partial S_0/\partial R \to -\infty$ at the value $R \to \infty$. In such a case, in the area higher than the axis of abscisses the ultimate curve is everywhere convex to it, while in the area lower than the axis of abscisses the ultimate curve has a point of inflection (in a general case, there is an odd number of such points).

For the curves, which sketch the permitted changes of R in the plane RS, we write down, according to (72),

$$\frac{\partial S}{\partial R} = Q\,. \tag{79}$$

For those points of these curves, which coincide with the points of the ultimate curve, we obtain, from (71) and (75), (70), (76) and (79),

$${}^*\dot{R}^2 = S_0 - S\,, \qquad {}^*\ddot{R} = \frac{1}{2}\left(\frac{\partial S_0}{\partial R} - \frac{\partial S}{\partial R}\right). \tag{80}$$

§16. Split the considered interval of time into the minimal number of the intervals of monotonic change of R. There on the opposite boundaries of each interval (such an interval can be finite or infinite) the quantity R has the minimal and the maximal numerical values along all the values attributed to R in it. We recognize four kinds of states conceivable for such a volume element at the minimal value of R: the kind m means the states of finite density at a regular minimum of R; the kind a means the states of finite density at an asymptotic value of R; the kind c means the states of infinitely high density at zero or finite speed of the change of R (in particular, this happens at the minimal or asymptotic value of R coinciding with R_∞); the kind s means the states of infinitely high density at the infinitely high speed of the change of R (these are so-called "special states"). We recognize also three kinds of states conceivable for such a volume element at the maximal value of R: the kind M means the states of finite density at a regular maximum of R; the kind A means the states of finite density at an asymptotic value of R; the kind D means the asymptotic states of zero density at $R \to \infty$.

The states m, a, M, A are attributed to all the points of the ultimate curve. The states D are attributed to all the points of an infinitely distant straight line $R = +\infty$, which lie not higher than the ultimate curve. As obvious, this is the whole straight line $R = +\infty$ in the case where $\Lambda > 0$, this is the half-line $R = +\infty$, $S \leqslant 0$ in the case where $\Lambda = 0$, and this is just a single point $R = +\infty$, $S = -\infty$ in the case where $\Lambda < 0$. The states c constitute just a point $R = R_\infty$, $S = +\infty$. The states c are attributed to all the points of the ultimate curve.

We denote each kind of evolution of R by a row of characters, which mean the states transited by a volume element with time along the time interval of the monotonic change of R. According to the notions, the kinds of evolution of a volume element, which are met in the theory of a homogeneous isotropic universe, should be recognized as follows: the kind A_1 as sA (expansion) or As (contraction); the kind A_2 as aD (expansion) or Da (contraction); the kind M_1 as sD (expansion) or Ds (contraction); the kind M_2 as DmD; the kind O_1 as sMs.

The conceivable kinds of evolution of a volume element in the intervals of the monotonic increase of R, permitted under different conditions, are given in the Table below.

Aside for the trivial case of the homogeneous isotropic models (2), the condition $Q = 0$ satisfies, for instance, at the centre of spherical symmetry in the absence of the power field. The condition $Q > 0$ satisfies, for instance, outside this centre and, in a general case, at all points where there is no power field, as well as no the absolute rotation, while

	$Q=0$	$Q \geqslant 0$	$Q \gtreqqless 0$
$\Lambda > 0$	sD, aD, mD sA sM	sD, aD, mD sA, aA, mA sM, aM, mM	sD, cD, aD, mD sA, cA, aA, mA sM, cM, aM, mM
$\Lambda = 0$	sD sM	sD sM	sD, cD, aD, mD sA, cA, aA, mA sM, cM, aM, mM
$\Lambda < 0$	 sM	 sM	sD, cD, aD, mD sA, cA, aA, mA sM, cM, aM, mM

the space deformation is anisotropic. The condition $Q<0$ satisfies, for instance, at the centre of spherical symmetry, which is the local centre of gravitational attraction in the sense of §11; and $Q<0$ satisfies also in the neighbourhood of such a centre of attraction.

§17. Here we provide some additional notes and comments to the previous results.

The solutions can have a physical meaning only outside the states of infinitely high density. It is meaningful to continue the solutions up to the states of infinitely high density. The formal conclusion about such states, obtained through the known equations of gravitation, should be considered, following Einstein, as a note on the inapplicability of these equations to the states of extremely high density such as the density inside atomic nuclei.

In the case of the homogeneous isotropic models, all the kinds shown in the Table are permitted. In the other cases, because we took into account not all of the equations of gravitation, we conclude that those kinds which are absent in this Table are impossible.

In the case of the homogeneous isotropic models, all that has been concluded about the evolution of any single volume element is also true for all elements of the considered three-dimensional region (which can be both finite and infinite). In a general case, these conclusions give a possibility to judge about the evolution of the rest volume elements, because of the continuity of space.

Along each interval of the monotonic change of R, all the rest quantities can be considered as functions of R in any case, not only in the case considered in §15. All the equations of §15 are true in the case of a barocline medium, which bear the viscosity of the 2nd kind, but is free of the 1st kind viscosity and the flows of energy. This allows us to distribute all the results of our Table onto this case, which is the most common for which this Table works.* In this Table we give only the permitted kinds of expansion. There are also the respective kinds of contraction corresponding to each of the kinds of expansion provided by this Table: the kind Ds corresponds to the kind sD, the kind Da corresponds to the kind aD, the kind Dm corresponds to the kind mD, and so forth. We consider the kinds of evolution of R in two adjacent intervals of the monotonic change, which are connected through a regular extremum of finite density. As is obvious, both kinds (expansion and contraction) should be in the row of the permitted kinds in all cases. However, in the case of a barotropic non-viscous medium, which is free of the flow of energy, and only in this case, we can assert that these two kinds are inverse to each other.

The behaviour of a homogeneous, isotropic model with time is valuable dependent on the Gaussian curvature of the space. In such a space, the numerical value of the Gaussian curvature is the same numerical value at all points, while the sign of the curvature remains unchanged with time, and is directly connected with the conditions of infiniteness of the space. In a general case, the correlation between the behaviour of a volume element and the Gaussian curvature is set up by the relations (68), (71) and (72). However there in the case of a homogeneous isotropic universe: 1) the Gaussian curvature changes from point to point, 2) it is impossible to assert that the sign of the Gaussian curvature remains unchanged at any point, 3) even if the space is holonomic, there is no direct connexion between the sign of the Gaussian curvature and the infiniteness of the space. In a general case, we should take into account the totality of the Riemannian curvatures at all points of the space, and along all two-dimensional directions in it (there in the homogeneous isotropic models they are everywhere equal to the Gaussian curvature). Using these curvatures, we are able to obtain the sufficient conditions of the infiniteness of a space, for instance

$$\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} \leqslant 0\,, \qquad \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} \geqslant 0\,, \quad B_{11} \leqslant 0\,, \tag{81}$$

where

$$B_{ik} = C_{ik} - \frac{1}{2}\,C h_{ik}\,. \tag{82}$$

*This is because that fact that the viscosity of the 1st kind and the flow of energy multiply the number of the allowed kinds of evolution.

It is possible to say that the simple statement of the problem about the infiniteness or the finiteness of space, which is specific to the theory of a homogeneous isotropic universe, is insufficient in the theory of such an inhomogeneous anisotropic universe whose space is holonomic, and is impossible in that case where the space is non-holonomic.

§18. In this Paragraph we consider a quasi-Newtonian approximation in cosmology, in the accompanying coordinates. In the framework of such an approximation, we use the equations of Newtonian mechanics (in Euclidean space) and Poisson's equation (or the generalization $\nabla_j^j \Phi = -4\pi\gamma\rho + \Lambda c^2$ of it, where Φ is the gravitational potential), without any universal ultimate conditions for the infiniteness of space. The reference frame accompanying the medium refines this approximation, because the velocity of macroscopic motions and some relativistic effects connected to it are zero in such coordinates. Therefore, the non-relativistic equations, constructed in the framework of the unitary interpretation of motion of a continuous medium [30], together with Poisson's equation (or its generalization given above) in the accompanying coordinates are both reasonable to be used as the quasi-Newtonian approximation to the chronometrically invariant relativistic equations.

Use the accompanying coordinates x^i and Newtonian mechanics in the pseudo-Euclidean space. Let t be Newtonian time. Let h_{ik}, h, D_{ik}, D, A_{ik} be the chronometrically invariant quantities which characterize the space of the accompanying frame of reference: the metric tensor, the fundamental determinant, the tensor of the rate of the space deformation, the speed of the relative volume expansion of the space, the tensor of angular velocity of the absolute rotation of the space. In such a quasi-Newtonian case,

$$D_{ik} = \frac{1}{2}\frac{\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{\partial h^{ik}}{\partial t}\,, \quad D = \frac{\partial \ln\sqrt{h}}{\partial t}\,, \tag{83}$$

$$\nabla_i \left(D_{jk} + A_{jk}\right) - \nabla_j \left(D_{ik} + A_{ik}\right) = 0\,, \tag{84}$$

$$\frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} + \frac{\partial A_{ij}}{\partial x^k} = 0\,. \tag{85}$$

Let F^k be the gravitational inertial force, acting per unit mass, which puts the surface forces into equilibrium. Let U^{ik} be the tensor of the density of the flow of momentum, while ρ is the density of mass. In such a case,

$$\nabla_i U^{ik} - \rho F^k = 0\,, \tag{86}$$

$$\frac{\partial\rho}{\partial t} + D\rho = 0\,, \qquad \frac{\partial}{\partial t}\left(\rho\sqrt{h}\right) = 0\,, \tag{87}$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{\partial F_k}{\partial x^i} - \frac{\partial F_i}{\partial x^k}\right) = 0\,, \tag{88}$$

$$\frac{\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D_k^j + A_{k\cdot}^{\cdot j}\right) + \frac{1}{2}\left(\nabla_i F_k + \nabla_k F_i\right) = \nabla_{ik}\Phi\,. \tag{89}$$

Contracting (84) term-by-term, we obtain

$$\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) = 0\,. \tag{90}$$

Contracting (89) term-by-term, while taking the equation of the potential into account, we obtain

$$\frac{\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + \nabla_j F^j = -\,4\pi\gamma\rho + \Lambda c^2. \tag{91}$$

It is obvious that the relativistic relations (14), (15), (16), the relativistic law of energy and momentum (33), (34), and the relativistic equations of gravitation (35), (36), (37) have the non-relativistic analogy in, respectively, the relations (85), (88), the equations (83) and (87), and the equations (86), (91), (90), (89). According to their physical meanings, (85) and (90) are identities like (84), (86) constitutes the equations of equilibrium, (87) is the continuity equation, (88) and (89) are the equations of motion of the medium, while (91), while taking (89) into account, substitutes instead the equation of the potential. These equations allow us to find the desired quasi-Newtonian approximation for the curvature. Comparing (37) and (89), we obtain

$$c^2 C_{ik} = DD_{ik} - D_{ij}D_k^j + 3A_{ij}A_{k\cdot}^{\cdot j} + \nabla_{ik}\Phi - \left(4\pi\gamma\rho + \Lambda c^2\right)h_{ik} \tag{92}$$

that leads to

$$c^2 C = D^2 - D_{jl}D^{jl} + 3A_{jl}A^{jl} - 16\pi\gamma\rho - 2\Lambda c^2. \tag{93}$$

As seen, in the framework of the quasi-Newtonian (non-relativistic) approximation, the equality (92) should be considered as the definition of the curvature tensor C_{ik}. At the same time, emphasizing the expansion of this formula by which comes the relativistic theory, we can calculate, through the equality (92) and its sequel (93), the Riemannian curvature and the Gaussian curvature of the accompanying space.

§19. Numerous researchers considered (and used) the similarity and analogy between the relativistic equations, obtained in the framework of different cosmological models, and the non-relativistic equations, obtained for the respective distribution and motion of masses. The first persons who did it were Milne and McCrea [31–33], who used this analogy for the homogeneous isotropic models, Bondi [18], who applied this analogy for the spherically symmetric models, and also Heckmann and Schücking [37], in the case of the axially symmetric homogeneous models (see also Heckmann [7], for this case). They all considered the cases with no pressure, viscosity and, flow of energy.* They removed Newtonian law of gravitation with a generalization of it, where the cosmological constant has been included. Such an application of Newtonian mechanics and Newtonian law of gravitation, based on the aforementioned analogy, is known as *Newtonian cosmology.* In such a cosmology, the uncertainty of the field of gravitation (the non-relativistic gravitational paradox) was either ignored or removed, in a hidden form, by some additional requirements, which are not usual in Newtonian theory. Neither the chronometric invariants in the relativistic equations nor the accompanying coordinates in the non-relativistic equations were applied by the aforementioned researchers. Almost all of them (see [34, 36, 37]) and also Layzer [35] discussed the question about the legitimacy of such a Newtonian cosmology. For instance, Heckmann and Schücking [38] supposed some changes on the ultimate conditions on the Newtonian potential at spatial infinity.

In contrast to the aforementioned authors, our method, which shows how to use this analogy (we proposed this method in §18), works in the framework of the following requirements:

1) Cancel any universal ultimate conditions on the potential at spatial infinity. (Considering every particular problem, such ultimate conditions or limitations used in the non-relativistic theory should meet analogous conditions or limitations assumed in the same problem considered in the relativistic theory);
2) Interpret the non-relativistic solutions as an approximation to the relativistic solutions. The use of the non-relativistic equations as the quasi-Newtonian approximation to the relativistic equations, includes the calculation for the space curvature;
3) Use of the chronometrically invariant quantities and operators in the relativistic equations. Such a use makes the relativistic equations look very similar to the non-relativistic equations;
4) Apply the accompanying coordinates in the non-relativistic equations. This makes the equations not only look similar to the rel-

*The presence of the factors leads to the lowering of the aforementioned analogy.

ativistic equations, but is also profitable to the quasi-Newtonian approximation itself;

5) Consideration of not only particular models, but also, and mainly, the general cases of relativistic cosmology.

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